

SYMMETRIC AND EXTERIOR POWERS OF CATEGORIES

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ABSTRACT. We define symmetric and exterior powers of categories, fitting into categorified Koszul complexes. We discuss examples and calculate the effect of these power operations on the categorical characters of matrix 2-representations.

INTRODUCTION

The classical formalism of symmetric and exterior powers in linear algebra plays a fundamental role in many areas of mathematics and physics: from geometry of Grassmann manifolds to supersymmetry and second quantization. In this paper we develop a “categorification” of this formalism. That is, we define symmetric and exterior powers not for vector spaces over a given field \mathbf{k} , but for \mathbf{k} -linear *categories*.

Tensor products of \mathbf{k} -linear categories have been defined in several contexts [KV94, Del90, Lyu01]. They all implement the following basic desideratum. Let A_1, A_2 be two associative \mathbf{k} -algebras, and $\mathcal{V}_i = A_i\text{-mod}$ be the category of left A_i -modules. Then the “categorical tensor product” $\mathcal{V}_1 \boxtimes \mathcal{V}_2$ should have something to do with the category $(A_1 \otimes_{\mathbf{k}} A_2)\text{-mod}$. Similarly for sheaves on spaces etc.

Given a workable concept of \boxtimes , the definition of the n th symmetric power of a category \mathcal{V} is rather straightforward: this is the category of S_n -invariant objects in $\mathcal{V}^{\boxtimes n}$. Here S_n is the symmetric group on n letters. In the physical language, this corresponds to considering “orbifold models” (with respect to S_n). In particular, let X be a smooth projective variety over \mathbb{C} and \mathcal{V} be the category of coherent sheaves on X . Taking the Grothendieck groups of the symmetric powers, we get the space

$$\mathcal{F} = \bigoplus_{n \geq 0} K(\mathrm{Sym}^n(\mathcal{V})) = \bigoplus_{n \geq 0} K^{S_n}(X^n)$$

which has the flavor of the loop Fock space. In fact, if we understand K as topological K-theory, then it is an observation of Grojnowski [Gro96] that \mathcal{F} is the irreducible representation of the loop Heisenberg algebra corresponding to $H^*(X, \mathbb{Z})$. This extends the results of [Gro96, Nak97] on the cohomology of the Hilbert schemes of a projective surface.

The categorical analog of exterior powers and the corresponding concept of “anti-equivariance” with respect to S_n , are less obvious. The key issue is what should play the role of the sign character $\mathrm{sgn} : S_n \rightarrow \{\pm 1\}$. There are (at least) two answers to this.

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Naive answer. The categorical analog of sgn should be an element of the second cohomology $H^2(S_n, \mathbf{k}^*)$, a group which was essentially described by I. Schur [Sch11] in his studies of projective representations of S_n . In fact, Schur’s results imply that for $n \geq 4$ and $\text{char}(\mathbf{k}) \neq 2$ this group contains exactly one nontrivial element c .

This point of view is motivated by the fact that the characters of any group G can be viewed as first homology classes: $\text{Hom}(G, \mathbf{k}^*) = H^1(G, \mathbf{k}^*)$. In physical language, it corresponds to considering “orbifold models with discrete torsion” [Di99]. Unfortunately, literal implementation of this point of view leads to a formalism which lacks some of the beauty and flexibility of the classical “ S - A -duality” (for example, Koszul complexes). In this paper we adopt the following

Super-algebra answer. The categorical analog of sgn should be a “Picard character” (see Definition 3.1.3)

$$\text{Sgn} : S_n \rightarrow \text{Pic}^{\mathbb{Z}/2}(\mathbf{k})$$

with values in the category of super-lines over \mathbf{k} . Here by a super-line we mean a super-vector space of super-dimension $(1|0)$ or $(0|1)$. Such a datum should combine both the classical character sgn and the 2-cocycle c as above.

This point of view draws on the wisdom accumulated in the theory of projective representations of S_n during the last 30 years [Joz89]: the theory becomes much more transparent if one introduces super-objects. Modern textbooks start with the super-approach right away [Kle05].

Using this as a guiding principle for our definition of categorical exterior powers, we find that several features of the classical theory generalize to the categorical setting. In particular, we have categorical analogs of Koszul complexes (§4) and we prove (Theorem 4.1.2) that on the level of complexified Grothendieck groups they give exact complexes of vector spaces.

The effect of categorical symmetric and exterior powers on Grothendieck groups does not at all reduce to taking ordinary symmetric and exterior powers of vector spaces. Rather, the generating functions for the dimensions of the Grothendieck groups of these powers lead to expressions involving the Euler function $\phi(q) = \prod_{n \geq 1} (1 - q^n)$ and remindful of the elliptic genus in topology. The reciprocity between such generating functions given by categorical Koszul complexes includes, in the simplest instance, the relation between $\phi(q)^{-1}$, the generating function for the numbers of partitions and $\prod_{n > 0} (1 + q^n)$, the generating function for the numbers of strict partitions of integers. In more sophisticated representation-theoretic examples (§5) the effect of categorical symmetric vs. exterior powers on Grothendieck groups can be expressed via the “logical proportion”

$$\frac{\text{Symmetric powers}}{\text{Exterior powers}} \sim \frac{\text{Untwisted Kac-Moody algebras}}{\text{Twisted Kac-Moody algebras}}.$$

There is a conceptual reason for the appearance of super-objects in projective representations of symmetric groups which, as far as we know, has not been recognized in representation-theoretical literature. It comes from the structure of the spherical spectrum \mathbb{S} in homotopy theory and from the Barratt-Priddy-Quillen (BPQ) theorem [Pri70, BP72] which reconstructs this spectrum from the symmetric groups. More precisely, we have the third (and truly fundamental) answer to the above question “what is the higher analog of the sign character”:

Homotopy-theoretic answer. There is a canonical homotopy theoretic character (coming from the BPQ theorem)

$$\mathbf{sgn} : S_n \longrightarrow \Omega\mathbb{S}_0$$

with values in the loop space of (one connected component of) the spherical spectrum. The truncation of this loop space in homotopy degrees 0 and 1 is described by the Picard category $\mathcal{P}ic^{\mathbb{Z}/2}(\mathbb{Z})$ formed by super-lines with integer structure. The Picard character \mathbf{Sgn} above is obtained from \mathbf{sgn} by this truncation:

$$S_n \xrightarrow{\mathbf{sgn}} \Omega\mathbb{S}_0 \rightsquigarrow \mathcal{P}ic^{\mathbb{Z}/2}(\mathbb{Z}) \longrightarrow \mathcal{P}ic^{\mathbb{Z}/2}(\mathbf{k}).$$

We explain this point of view in more detail in §3.1.9. As one can take less drastic truncations of $\Omega\mathbb{S}_0$, this provides a systematic way of defining 2-categorical, 3-categorical etc. analogs of the sign character. For instance one could take the truncation of $\Omega\mathbb{S}_0$ in degrees $[0, 2]$ and interpret the resulting homotopy object in terms of a Picard 2-category, thus getting a “2-categorical sign character” from first principles.

The paper is organized as follows. In §1 we discuss various approaches to defining tensor products of \mathbf{k} -linear categories: abelian, pre-triangulated etc. With the eye on later applications, we also introduce the concept of a super-linear category and discuss Grothendieck groups in this context. Symmetric powers of categories are defined and studied in §2. In §3 we introduce exterior powers, starting from an extended discussion in (3.1) of what should play the role of the sign character. We consider first the “naive” approach and then correct it using the super point of view. In §4 we introduce the categorical Koszul complexes and prove that they lead to exact complexes of complexified Grothendieck groups. Further, §5 is devoted to examples of symmetric and exterior powers coming from representation theory. We show how they give companion Kac-Moody-type objects, one “untwisted”, the other “twisted”. In §6 we study symmetric and exterior powers on the category of 2-representations [GK07] of a group G and find the effect of these constructions on 2-characters of such 2-representations. Finally, §7 is devoted to discussion of further directions and open question.

1. TENSOR PRODUCTS OF CATEGORIES

(1.1) Definition of tensor products and direct sums. Let \mathbf{k} be an algebraically closed field. By a linear category we will always mean an additive \mathbf{k} -linear category, i.e., a category \mathcal{V} with finite direct sums, in which all the sets $\text{Hom}_{\mathcal{V}}(V, W)$ are made into \mathbf{k} -vector spaces so that the composition of morphisms is bilinear.

Let \mathcal{V}, \mathcal{W} be two linear categories. Following [KV94], §5.18 (see also [BK01], Def. 1.1.15), we define their tensor product $\mathcal{V} \boxtimes \mathcal{W}$ to have, as objects, the symbols (formal direct sums of formal tensor products)

$$(1.1.1) \quad \bigoplus_{i=1}^n V_i \boxtimes W_i, \quad V_i \in \text{Ob}(\mathcal{V}), W_i \in \text{Ob}(\mathcal{W}).$$

Morphisms between such symbols are defined as follows. First, we consider the case when the formal sums in (1.1.1) have only one summand. In this case we put

$$(1.1.2) \quad \text{Hom}_{\mathcal{V} \boxtimes \mathcal{W}}(V \boxtimes W, V' \boxtimes W') = \text{Hom}_{\mathcal{V}}(V, V') \otimes_{\mathbf{k}} \text{Hom}_{\mathcal{W}}(W, W').$$

Then, we define

$$(1.1.3) \quad \text{Hom}_{\mathcal{V} \boxtimes \mathcal{W}} \left(\bigoplus_{j=1}^m V_j \boxtimes W_j, \bigoplus_{i=1}^n V'_i \boxtimes W'_i \right) = \left\| \text{Hom}_{\mathcal{V} \boxtimes \mathcal{W}}(V_j \boxtimes W_j, V'_i \boxtimes W'_i) \right\|_{j=1, \dots, m}^{i=1, \dots, n}$$

to consist of n by m matrices with the (i, j) th matrix element being a morphism from the j th formal summand of the source to the i th formal summand of the target. As this definition of morphisms with matrices mimics morphisms between direct sums, we get that (1.1.1) is indeed a categorical direct sum of the $V_i \boxtimes W_i$ in $\mathcal{V} \boxtimes \mathcal{W}$. Further, notice the following:

(1.1.4) Proposition. Let V_1, V_2 be objects of \mathcal{V} . Then

$$(V_1 \oplus V_2) \boxtimes W = (V_1 \boxtimes W) \oplus (V_2 \boxtimes W),$$

i.e., the left-hand side is a categorical direct sum, in $\mathcal{V} \boxtimes \mathcal{W}$, of the two summands in the right-hand side. Similarly for $V \boxtimes (W_1 \oplus W_2)$.

Proof. In any additive category \mathcal{C} , to say that an object X is a categorical direct sum of X_1 and X_2 , is the same as to say that there are morphisms in \mathcal{C}

$$i_\nu : X_\nu \rightarrow X, \quad p_\nu : X \rightarrow X_\nu, \quad \nu = 1, 2,$$

such that

$$p_\nu i_\nu = \text{Id}_{X_\nu}, \quad p_\nu i_\mu = 0, \quad \nu \neq \mu, \quad i_1 p_1 + i_2 p_2 = \text{Id}_X.$$

Suppose we have such morphisms for $\mathcal{C} = \mathcal{V}$, $X_\nu = V_\nu$ and $X = V_1 \oplus V_2$. Then tensoring them with Id_W in (1.2), we get the required morphisms for $\mathcal{C} = \mathcal{V} \boxtimes \mathcal{W}$ and for the objects claimed in the proposition. \square

We also use the notation $\mathcal{V} \boxplus \mathcal{W}$ for the direct sum (Cartesian product) of two linear categories, i.e.,

$$(1.1.5) \quad \begin{aligned} \text{Ob}(\mathcal{V} \boxplus \mathcal{W}) &= \text{Ob}(\mathcal{V}) \times \text{Ob}(\mathcal{W}), \\ \text{Hom}_{\mathcal{V} \boxplus \mathcal{W}}((V, W), (V', W')) &= \text{Hom}_{\mathcal{V}}(V, V') \oplus \text{Hom}_{\mathcal{W}}(W, W'). \end{aligned}$$

(1.1.6) Proposition. For any three linear categories $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}$ we have a natural equivalence

$$(\mathcal{V}_1 \boxplus \mathcal{V}_2) \boxtimes \mathcal{W} = (\mathcal{V}_1 \boxtimes \mathcal{W}) \boxplus (\mathcal{V}_2 \boxtimes \mathcal{W}).$$

Similarly for $\mathcal{V} \boxtimes (\mathcal{W}_1 \boxplus \mathcal{W}_2)$.

Proof. A functor from the left-hand side to the right-hand side is given by

$$\Phi : (V_1, V_2) \boxtimes W \mapsto (V_1 \boxtimes W, V_2 \boxtimes W).$$

A functor from the right-hand side to the left-hand side is given by

$$\Psi : (V_1 \boxtimes W_1, V_2 \boxtimes W_2) \mapsto ((V_1, 0) \boxtimes W_1) \oplus ((0, V_2) \boxtimes W_2).$$

The fact that Φ and Ψ are quasi-inverse to each other follows easily from Proposition 1.1.4. \square

(1.2) Examples. We now list some immediate examples.

(1.2.1) Algebras. For a \mathbf{k} -algebra A we denote by $A\text{-mod}$ the category of left A -modules and by $A\text{-mod}^f$ the subcategory of finitely presented left modules. Similarly, we write $\text{mod-}A$ and $\text{mod}^f\text{-}A$ for the corresponding right module categories. For any two algebras A and B we have a fully faithful embedding

$$(1.2.1)(a) \quad (A\text{-mod}^f) \boxtimes (B\text{-mod}^f) \longrightarrow (A \otimes_{\mathbf{k}} B)\text{-mod}^f$$

$$\bigoplus_{i=1}^n M_i \boxtimes N_i \longmapsto \bigoplus_{i=1}^n M_i \otimes_{\mathbf{k}} N_i,$$

and similarly for right-modules. We also have an equivalence

$$(1.2.1)(b) \quad (A\text{-mod}^f) \boxplus (B\text{-mod}^f) \longrightarrow (A \oplus B)\text{-mod}^f$$

$$(M, N) \longmapsto M \oplus N,$$

and the analogous equivalence for right-module categories. If A and B are semisimple then the embedding in (1.2.1)(a) is an equivalence of categories.

(1.2.2) 2-vector spaces. By a 2-vector space we will mean a semisimple abelian \mathbf{k} -linear category with finitely many isomorphism classes of simple objects, cf. [KV94]. Thus, if A is semisimple, then $A\text{-mod}^f$ is a 2-vector space. We will denote by $[n] = \mathbf{k}^n\text{-mod}$ the standard n -dimensional 2-vector space. Each 2-vector space \mathcal{V} is equivalent to $[n]$, where n is the number of simple objects in \mathcal{V} up to isomorphism. We write $n = \text{Dim}(\mathcal{V})$. Thus, we have

$$[n] \boxtimes [m] = [nm], \quad [n] \boxplus [m] = [n + m].$$

(1.2.3) Coherent sheaves. Let X be a \mathbf{k} -scheme of finite type. Denote by Coh_X the category of coherent sheaves of \mathcal{O}_X -modules. For any two schemes X, Y as above we have a fully faithful embedding

$$(1.2.3)(a) \quad \text{Coh}_X \boxtimes \text{Coh}_Y \longrightarrow \text{Coh}_{X \times Y}$$

$$\mathcal{F} \boxtimes \mathcal{G} \longmapsto p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G},$$

where p_X, p_Y are the projections of $X \times Y$ to X and Y , respectively. We also have an equivalence

$$(1.2.3)(b) \quad \text{Coh}_X \boxplus \text{Coh}_Y \longrightarrow \text{Coh}_{X \sqcup Y},$$

where $X \sqcup Y$ is the disjoint union. If $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine, this reduces to (1.2.1).

We denote by $D^b(X) = D^b(\operatorname{Coh}_X)$ the bounded derived category of coherent sheaves on X . By passing to the derived functors in (1.2.3), we get a fully faithful embedding and an equivalence

$$(1.2.3)(c) \quad \begin{aligned} D^b(X) \boxtimes D^b(Y) &\longrightarrow D^b(X \times Y) \\ D^b(X) \boxplus D^b(Y) &\longrightarrow D^b(X \sqcup Y). \end{aligned}$$

(1.2.4) Characterization by a (2-)universal property. Let \mathcal{X} be another additive \mathbf{k} -linear category. A functor

$$\psi : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{X}$$

is called bilinear, if it is linear in each of the arguments. This implies, as in Proposition 1.1.4, that it takes direct sums in each argument to direct sums. For example, the functor

$$\begin{aligned} \boxtimes : \mathcal{V} \times \mathcal{W} &\longrightarrow \mathcal{V} \boxtimes \mathcal{W} \\ (V, W) &\longmapsto V \boxtimes W \end{aligned}$$

is bilinear. Moreover, it is the universal bilinear functor, which means that for any \mathcal{X} , precomposition with \boxtimes defines an equivalence

$$\mathcal{F}\operatorname{un}_{\mathbf{k}}(\mathcal{V} \boxtimes \mathcal{W}, \mathcal{X}) \xrightarrow{\simeq} \mathcal{F}\operatorname{un}_{\mathbf{k}\text{-bil}}(\mathcal{V} \times \mathcal{W}, \mathcal{X})$$

(categories of (bi-)linear functors and linear natural transformations). This universal property, noticed in [BK01], characterizes the category $\mathcal{V} \boxtimes \mathcal{W}$ uniquely up to an equivalence of categories, which is unique up to a unique isomorphism.

(1.3) Completed tensor products: abelian case. If the categories \mathcal{V}, \mathcal{W} in (1.1) are abelian (resp. triangulated), then $\mathcal{V} \boxtimes \mathcal{W}$ is, in general, no longer abelian (resp. triangulated). One would like to enlarge $\mathcal{V} \boxtimes \mathcal{W}$ to a bigger abelian (resp. triangulated) category $\mathcal{V} \widehat{\boxtimes} \mathcal{W}$ which would include kernels/cokernels (resp. cones) of morphisms between objects of the form (1.1.1).

Assume \mathcal{V}, \mathcal{W} abelian, and let \mathcal{X} be another abelian \mathbf{k} -linear category. We can then speak about bilinear functors

$$\Psi : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{X},$$

which are bi-right-exact, i.e., carry cokernels in each of the variables into cokernels in \mathcal{X} . The abelian tensor product of Deligne [Del90] is an abelian category $\mathcal{V} \widehat{\boxtimes} \mathcal{W}$ together with a bi-right-exact functor

$$(1.3.1) \quad \boxtimes : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \widehat{\boxtimes} \mathcal{W}.$$

satisfying the following (2-)universal property: for any other abelian category \mathcal{X} , precomposition with \boxtimes defines an equivalence

$$\mathcal{F}\operatorname{un}_{\mathbf{k}, \text{r.e.}}(\mathcal{V} \widehat{\boxtimes} \mathcal{W}, \mathcal{X}) \xrightarrow{\simeq} \mathcal{F}\operatorname{un}_{\mathbf{k}\text{-bil}, \text{r.e.}}(\mathcal{V} \times \mathcal{W}, \mathcal{X}).$$

Here $\mathcal{F}\operatorname{un}_{\mathbf{k}(-\text{bil}), \text{r.e.}}$ denotes the category of (bi-)linear (bi-)right exact functors and linear natural transformations. These properties characterize $\mathcal{V} \widehat{\boxtimes} \mathcal{W}$, if it exists, uniquely up to an equivalence of categories, unique up to a unique isomorphism.

(1.3.2) Examples. (a) ([Del90], Prop. 5.3) Let A and B be right-coherent \mathbf{k} -algebras. Then

$$(\text{mod}^f\text{-}A) \hat{\boxtimes} (\text{mod}^f\text{-}B) \simeq \text{mod}^f\text{-}(A \otimes_{\mathbf{k}} B).$$

(b) ([Del90], Prop. 5.11) Let A be a finite-dimensional \mathbf{k} -algebra and \mathcal{V} any abelian \mathbf{k} -linear category. Then $(\text{mod}^f\text{-}A) \hat{\boxtimes} \mathcal{V}$ exists and is equivalent to the category of \mathbf{k} -linear right A -modules in \mathcal{V} , i.e., objects $V \in \mathcal{V}$ together with a k -algebra homomorphism $A^{op} \rightarrow \text{End}_{\mathcal{V}}(V)$.

(c) [Lyu01] Let (X, \mathcal{S}) be a stratified space, so \mathcal{S} is the set of strata, partially ordered by inclusion of the closures. A perversity function for (X, \mathcal{S}) is a monotone function $p : \mathcal{S} \rightarrow \mathbb{Z}$. To this data one associates an abelian category $\text{Perv}(X, \mathcal{S}, p)$ of p -perverse sheaves of \mathbf{k} -vector spaces on X relative to \mathcal{S} . If (X', \mathcal{S}') is another stratified space and p' a perversity function for it then the product $X \times X'$ is stratified by the products of the strata, who form the set $\mathcal{S} \times \mathcal{S}'$. It has the perversity function $p \dot{+} p'$ given by

$$(p \dot{+} p')(S \times S') = p(S) + p'(S'), \quad S \in \mathcal{S}, S' \in \mathcal{S}'.$$

In this situation, we have

$$\text{Perv}(X, \mathcal{S}, p) \hat{\boxtimes} \text{Perv}(X', \mathcal{S}', p') = \text{Perv}(X \times X', \mathcal{S} \times \mathcal{S}', p \dot{+} p').$$

(1.4) Completed tensor products: pre-triangulated case. Let \mathcal{V}, \mathcal{W} , and \mathcal{X} be \mathbf{k} -linear triangulated categories. Then one can speak about bilinear functors $\Psi : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{X}$ which are bi-exact, i.e., take exact triangles in each of the variables into exact triangles in \mathcal{X} . A natural approach to the definition of the “triangulated tensor product” of \mathcal{V} and \mathcal{W} would be to look for the target of the universal bi-exact functor. However, as pointed out in [BLL], it is difficult to prove existence in any nontrivial case, as one does not really have control over bi-exact functors. An alternative approach, proposed in [BLL], works with *enhanced triangulated categories*, following [BK90].

(1.4.1) Definition. By a dg-category we mean a category \mathcal{V} enriched in the monoidal category of complexes of \mathbf{k} -vector spaces. Thus for any $V, V' \in \text{Ob}(\mathcal{V})$ we have a complex $\text{Hom}_{\mathcal{V}}^{\bullet}(V, V')$. We denote by $H^0(\mathcal{V})$ the \mathbf{k} -linear category with the same objects of \mathcal{V} and

$$\text{Hom}_{H^0(\mathcal{V})}(V, V') = H^0 \text{Hom}_{\mathcal{V}}(V, V').$$

A dg-functor between dg-categories is a functor preserving the enrichment, i.e., inducing morphisms of Hom-complexes. A dg-functor $F : \mathcal{V} \rightarrow \mathcal{V}'$ is called a quasi-equivalence, if $H^{\bullet}(F) : H^{\bullet}(\mathcal{V}) \rightarrow H^{\bullet}(\mathcal{V}')$ is an equivalence of categories.

(1.4.2) Example. If \mathcal{A} is an abelian \mathbf{k} -linear category, then $C(\mathcal{A})$, the category of cochain complexes over \mathcal{A} , is a dg-category (see [BK90, p.94]). The category $H^0 C(\mathcal{A})$ is the homotopy category of complexes.

We write $C(\mathbf{k}\text{-mod})$ for the dg-category of complexes of \mathbf{k} -vector spaces. If \mathcal{V} and \mathcal{W} are two dg-categories, then the dg-functors from \mathcal{V} to \mathcal{W} form themselves the objects of a dg-category $\mathcal{F}\text{un}_{\text{dg}}(\mathcal{V}, \mathcal{W})$ (see [BK90, p.95]) and the opposite category \mathcal{V}^{op} is again a dg-category. For a dg-category \mathcal{V} , we define

$$(1.4.3) \quad \text{mod-}\mathcal{V} := \mathcal{F}\text{un}_{\text{dg}}(\mathcal{V}^{\text{op}}, C(\mathbf{k} - \text{mod})).$$

This is a dg-category with shift and cone functors, inherited from $C(k\text{-mod})$. The original dg-category \mathcal{V} is realized as a full dg-subcategory of $\text{mod-}\mathcal{V}$ via the Yoneda embedding.

(1.4.4) Definition. Let $\text{Pre-Tr}(\mathcal{V})$ be the smallest full dg-subcategory of $\text{mod-}\mathcal{V}$ that contains \mathcal{V} and is closed under isomorphisms, direct sums, shifts and cones. Let $\text{Perf}(\mathcal{V})$ be the full dg-subcategory of $\text{mod-}\mathcal{V}$ consisting of semi-free¹ dg-modules that are homotopy equivalent to a direct summand of an object of $\text{Pre-Tr}(\mathcal{V})$.

Bondal and Kapranov have given a hands-on description of the category $\text{Pre-Tr}(\mathcal{V})$:

(1.4.5) Proposition. The category $\text{Pre-Tr}(\mathcal{V})$ is equivalent to the category of *one sided twisted complexes*² of [BK90, Sec.4].

Proof. In [loc.cit], Bondal and Kapranov prove that the category of one-sided twisted complexes may be embedded as a full subcategory in $\text{mod-}\mathcal{V}$ in such a way that it contains \mathcal{V} and is closed under direct sums, shifts and cones. It follows that this category contains $\text{Pre-Tr}(\mathcal{V})$. To see equality, one notes that every one-sided twisted complex can be obtained from objects in \mathcal{V} by taking successive cones of degree zero morphisms (see [BLL, Prop.3.10]). \square

We denote

$$(1.4.6) \quad \text{Tr}(\mathcal{V}) = H^0 \text{Pre-Tr}(\mathcal{V}).$$

(1.4.7) Definition. A dg-category \mathcal{V} is called *pre-triangulated*, if the embedding

$$i_{\mathcal{V}}: \mathcal{V} \longrightarrow \text{Pre-Tr}(\mathcal{V})$$

is a quasi-equivalence. We say that \mathcal{V} is *perfect*, if $\mathcal{V} \rightarrow \text{Perf}(\mathcal{V})$ is a quasi-equivalence.

Note that “perfect” implies “pre-triangulated”. If \mathcal{V} is pre-triangulated, then $H^0(\mathcal{V})$ is naturally a triangulated category. Given a triangulated category \mathcal{D} , an enhancement of \mathcal{D} is a pre-triangulated category \mathcal{V} with an equivalence

$$\epsilon: H^0(\mathcal{V}) \rightarrow \mathcal{D}$$

of triangulated categories.

(1.4.8) Examples. (a) If \mathcal{V} is any dg-category, then $\text{Pre-Tr}(\mathcal{V})$ is pre-triangulated, and hence $\text{Tr}(\mathcal{V})$ is triangulated.

(b) If \mathcal{A} is any \mathbf{k} -linear abelian category, then $C(\mathcal{A})$, the category of complexes over \mathcal{A} , is pre-triangulated.

(c) Let X be a \mathbf{k} -scheme of finite type. Then the triangulated category $D^b(X)$ has the following enhancement $I(X)$. Let $\mathcal{O}_X\text{-mod}$ be the abelian category of all sheaves of \mathcal{O}_X -modules (not necessarily quasicohherent). Then $I(X)$ is the full dg-subcategory in $C(\mathcal{O}_X\text{-mod})$ consisting of complexes of injective objects, which are bounded below and have only finitely many cohomology sheaves, all coherent. The dg-category $I(X)$ is perfect. It is well known that $H^0 I(X)$, i.e., the homotopy category of complexes as above, is equivalent to $D^b(X)$.

The following is an equivalent reformulation of [BLL], Def. 4.2.

¹An object F of $\text{mod-}\mathcal{V}$ is *semi-free* if it has a filtration $0 = F_0 \subset F_1 \subset \dots = F$, such that F_{i+1}/F_i is isomorphic to a direct sum of shifted free objects, see [BLL, p.11].

²In [BK90], the category of one sided twisted complexes is called $\text{Pre-Tr}^+(\mathcal{V})$.

(1.4.9) Definition. Let \mathcal{V}, \mathcal{W} be perfect dg-categories. Their completed tensor product is defined by

$$\mathcal{V} \widehat{\boxtimes} \mathcal{W} = \text{Perf}(\mathcal{V} \boxtimes \mathcal{W}),$$

where $\mathcal{V} \boxtimes \mathcal{W}$ is defined as in (1.1), using tensor products of complexes and composition with a sign:

$$(f \otimes g) \circ (h \otimes k) := (-1)^{\deg(g) \cdot \deg(h)} (fh) \otimes (gk).$$

(1.4.10) Example. ([BLL], Th. 5.5) Let X, Y be smooth projective varieties over \mathbf{k} . Then $I(X) \widehat{\boxtimes} I(Y)$ is quasi-equivalent to $I(X \times Y)$.

There is no obvious (2-)universal property characterizing completed tensor products of perfect dg-categories.

(1.5) Super-objects. 2-periodic case. Let $(\text{SVect}^f, \otimes)$ be the symmetric monoidal category of finite dimensional super-vector spaces over \mathbf{k} , see [Man92] [Kle05]. Thus, objects of SVect^f are finite dimensional $\mathbb{Z}/2$ -graded vector spaces $V = V_0 \oplus V_1$, morphisms are linear operators preserving grading, and \otimes is the usual tensor product,

$$(V \otimes W)_{\bar{0}} = V_0 \otimes W_0 \oplus V_1 \otimes W_1,$$

$$(V \otimes W)_{\bar{1}} = V_1 \otimes W_0 \oplus V_0 \otimes W_1,$$

with the symmetry isomorphism given by the Koszul sign rule,

$$\begin{aligned} V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto (-1)^{\deg(v) \cdot \deg(w)} w \otimes v. \end{aligned}$$

The shift of $\mathbb{Z}/2$ -grading will be denoted by Π , so

$$(1.5.1) \quad (\Pi V)_{\bar{0}} = V_1, \quad (\Pi V)_{\bar{1}} = V_0.$$

(1.5.2) Definition. By a super-linear category we will mean a \mathbf{k} -linear category which is a module category over $(\text{SVect}^f, \otimes)$.

Thus, in a super-linear category \mathcal{V} we have the functor Π given by

$$(1.5.3) \quad \Pi(V) = \Pi(k) \otimes V,$$

and each $\text{Hom}_{\mathcal{V}}(V, V')$ is naturally extended to a super-vector space with

$$(1.5.4) \quad \text{Hom}_{\bar{0}}(V, W) = \text{Hom}_{\mathcal{V}}(V, W), \quad \text{Hom}_{\bar{1}}(V, W) = \text{Hom}_{\mathcal{V}}(V, \Pi W).$$

We will denote by \mathcal{V}_{\bullet} the *graded extension* of \mathcal{V} , i.e., the category with the same objects as \mathcal{V} and the extended ($\mathbb{Z}/2$ -graded) Hom-sets above. We will refer to \mathcal{V} as the *even part* of \mathcal{V}_{\bullet} . An irreducible object V of a superlinear category \mathcal{V} is called *self-associate* if V is isomorphic to ΠV . Otherwise, V is called *absolutely irreducible*.

(1.5.5) Examples. (a) Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a finite dimensional associative and unitary superalgebra, i.e., an associative and unitary algebra in SVect^f . A right A -supermodule M is called *finitely generated* if there is an even surjective morphism

$$A^{\oplus l} \oplus \Pi(A)^{\oplus k} \twoheadrightarrow M$$

of right A -supermodules, and M is called *finitely presented* if it fits into an exact sequence of even morphisms

$$A^{\oplus i} \oplus \Pi(A)^{\oplus j} \longrightarrow A^{\oplus l} \oplus \Pi(A)^{\oplus k} \longrightarrow M \longrightarrow 0.$$

We write $\text{Smod}^f\text{-}A$ for the category of finitely presented right A -supermodules and even morphisms. This is a superlinear category. If A is finitely generated then $\text{Smod}^f\text{-}A$ is an abelian superlinear category.

(b) Consider the Clifford superalgebras

$$C_1 = \mathbf{k}[\xi]/(\xi^2)$$

and

$$C_2 = C_1 \otimes C_1 \cong \mathbf{k}[\xi_1, \xi_2]/(\xi_1^2, \xi_2^2, \xi_1\xi_2 - \xi_2\xi_1),$$

see [Kle05, (12.1.3), (12.2.4)]. In $\text{Smod}^f\text{-}C_1$ we have exactly one isomorphism class of irreducible objects, namely that of C_1 viewed as right-module over itself. In particular, C_1 is self-associate. In $\text{Smod}^f\text{-}C_2$, we have a decomposition

$$C_2 \cong U \oplus \Pi U,$$

where U and ΠU are non-isomorphic irreducible objects, and any irreducible object is isomorphic to one of these two. It follows that we have an equivalence of superlinear categories

$$\text{Smod}^f\text{-}C_2 \simeq \text{SVect}^f.$$

(c) Let now \mathcal{V} be a superlinear 2-vectorspace with $2n$ isomorphism classes of absolutely irreducible objects (these come in pairs $U, \Pi U$) and m isomorphism classes of self-associate irreducible objects. Then \mathcal{V} is equivalent, as a superlinear category, to the “*standard super 2-vectorspace*” of these dimensions,

$$[n|m] := (\text{SVect}^f)^n \boxplus (\text{Smod}^f\text{-}C_1)^m.$$

(d) Let \mathcal{A} be an abelian \mathbf{k} -linear category. Let $C^{(2)}(\mathcal{A})$ be the category of 2-periodic complexes over \mathcal{A} , i.e., of complexes (V^\bullet, d) such that $V^i = V^{i+2}$ and $d_i = d_{i+2}$ for all i . Such complexes can be equally well considered $\mathbb{Z}/2$ -graded. Morphisms in $C^{(2)}(\mathcal{A})$ are also assumed to be 2-periodic. Then $C^{(2)}(\mathcal{A})$ as well as the homotopy category $H^0 C^{(2)}(\mathcal{A})$ are super-linear categories with Π induced by the shift of grading of complexes.

(e) Let \mathcal{V} be a 2-periodic dg-category, i.e., each complex $\text{Hom}_{\mathcal{V}}^\bullet(V, V')$ is 2-periodic. Then \mathcal{V}^{op} and the category

$$\text{mod}^{(2)}\text{-}\mathcal{V} := \mathcal{F}\text{un}_{\text{dg}}(\mathcal{V}^{\text{op}}, C^{(2)}(\mathbf{k}\text{-mod}))$$

of contravariant functors into even periodic cochain complexes are again 2-periodic. So is $\text{Pre-Tr}^{(2)}(\mathcal{V})$, the smallest full subcategory of $\text{mod}^{(2)}\text{-}\mathcal{V}$ that contains \mathcal{V} and is closed under isomorphisms, direct sums, shifts and cones.

For each object C we have $C[2] = C$, so the even subcategories of $\text{Pre-Tr}^{(2)}(\mathcal{V})$ and $\text{Perf}^{(2)}(\mathcal{V})$ are super-linear with $\Pi C = C[1]$. Note that this definition of Π makes sense in $\text{Pre-Tr}^{(2)}(\mathcal{V})$, but that Π might not be well-defined in \mathcal{V} , even if the latter is pre-triangulated. In the following, it is understood that the term ‘*2-periodic pre-triangulated (or perfect) dg-category*’ refers to a category \mathcal{V} of this kind where the objects $C[1]$ exist in \mathcal{V} . Hence we may speak of the even subcategory of \mathcal{V} as a superlinear category with graded extension \mathcal{V} (rather than having to pass to $\text{Pre-Tr}^{(2)}(\mathcal{V})$).

(1.6) Superlinear functors and supernatural transformations. Let \mathcal{V} and \mathcal{W} be superlinear categories. A superlinear functor from \mathcal{V} to \mathcal{W} consists of a linear functor $F: \mathcal{V} \rightarrow \mathcal{W}$ together with a linear natural isomorphism $\phi: \Pi F \cong F \Pi$, satisfying

$$(\phi \Pi) \circ (\Pi \phi) = \text{Id}_F.$$

Given two such pairs, (F, ϕ) and (H, ψ) , a *supernatural transformation* between them is a linear natural transformation $\xi: F \Rightarrow H$ satisfying

$$\psi \circ (\Pi \xi) = (\xi \Pi) \circ \phi.$$

We will denote the set of supernatural transformations from (F, ϕ) to (H, ψ) by

$$\text{sNat}((F, \phi), (H, \psi)),$$

occasionally dropping ϕ and ψ from the notation. Further, we will write

$$\text{sTr}(F, \phi) := \text{sNat}((\text{Id}, \text{Id}), (F, \phi))$$

and

$$\text{sCenter}(\mathcal{V}) := \text{sTr}(\text{Id}, \text{Id}).$$

(1.6.1) Examples. (a) We have

$$\text{sCenter}(\text{SVect}^f) \cong \mathbf{k}.$$

This includes diagonally into

$$\mathcal{N}\text{at}(\text{Id}, \text{Id}) \cong \mathbf{k}^2.$$

(b) Similarly,

$$\text{sTr}(\text{Id}, -\text{Id}) \cong \mathbf{k},$$

but now the inclusion into $\mathcal{N}\text{at}(\text{Id}, \text{Id})$ is the skew diagonal map $a \mapsto (a, -a)$. This difference is picked up by the action of Π : in the first case we have $\Pi \xi \Pi = \xi$, and in the second case we have $\Pi \xi \Pi = -\xi$.

(c) More generally, let \mathcal{V} be a superlinear 2-vectorspace, and let n be the number of isomorphism classes, up to shift of grading, of irreducible objects in \mathcal{V} . Then, by Schur's Lemma,

$$\text{sTr}(\text{Id}, \text{Id}) \cong \mathbf{k}^n,$$

while the \mathbf{k} -dimension of

$$\text{sTr}(\Pi, \text{Id})$$

counts the isomorphism classes of self-associate objects of \mathcal{V} .

(1.6.2) Definition. We will write $\text{sFun}_{\mathbf{k}}(\mathcal{V}, \mathcal{W})$ for the category of superlinear functors from \mathcal{V} to \mathcal{W} and supernatural transformations between them. This is itself a superlinear category with shift functor

$$\Pi(H, \phi) := (\Pi H, -\Pi \phi).$$

We have a superlinear equivalence of categories

$$(1.6.3) \quad \begin{array}{ccc} \text{SVect}^f & \longrightarrow & \text{sFun}_{\mathbf{k}}(\text{SVect}^f, \text{SVect}^f) \\ V_{\bullet} & \longmapsto & (V_{\bullet} \otimes -, \phi_{V_{\bullet}}), \end{array}$$

where $\phi_{V_{\bullet}}$ is given by the symmetry isomorphism (with Koszul sign)

$$\phi_{V_{\bullet}}: \Pi k \otimes V_{\bullet} \cong V_{\bullet} \otimes \Pi k.$$

Superlinear functors extend canonically to even \mathbf{k} -linear functors between the extended categories, and we have an equivalence of categories

$$(1.6.4) \quad \mathbf{sFun}_{\mathbf{k}}(\mathcal{V}, \mathcal{W}) \xrightarrow{\simeq} \mathcal{F}\mathbf{un}_{\mathbf{k}, \text{ev}}(\mathcal{V}_{\bullet}, \mathcal{W}_{\bullet}),$$

where $\mathcal{F}\mathbf{un}_{\mathbf{k}, \text{ev}}$ stands for the category of \mathbf{k} -linear functors that preserve the degree of morphisms and \mathbf{k} -linear natural transformations consisting of only even maps.

Proof. Indeed, a supernatural transformation between two superlinear functors is the same thing as an even \mathbf{k} -linear natural transformation between the extended functors. Further, let $F: \mathcal{V}_{\bullet} \rightarrow \mathcal{W}_{\bullet}$ be an even \mathbf{k} -linear functor, and let $\text{Id}^{\flat}: \Pi \Rightarrow \text{Id}$ be the natural isomorphism consisting of the identities, viewed as odd morphisms $\Pi V \rightarrow V$. Then

$$F(\text{Id}^{\flat}): F\Pi \Longrightarrow F$$

is an odd natural transformation, and we set $\phi := F(\text{Id}^{\flat})$, viewed as an even natural transformation from $F\Pi$ to ΠF . Then the pair $(F|_{\mathcal{V}}, \phi)$ is a superlinear functor, whose graded extension equals F . \square

The composite of two super-linear functors is again a super-linear functor in a canonical way, and composition is compatible with forming the extended functors.

(1.6.5) Tensor products of superlinear categories: uncompleted case. Let \mathcal{V} and \mathcal{W} be super-linear categories. Then the categorical tensor product of their extended categories is canonically enriched over \mathbf{SVect}^f . Let

$$\mathcal{V} \boxtimes_s \mathcal{W} := (\mathcal{V}_{\bullet} \boxtimes \mathcal{W}_{\bullet})_{\text{ev}}.$$

Note that the inclusion

$$B: \mathcal{V} \boxtimes \mathcal{W} \longrightarrow \mathcal{V} \boxtimes_s \mathcal{W}$$

is faithful and essentially surjective, but not full. In particular, in $\mathcal{V} \boxtimes_s \mathcal{W}$ we have the functor isomorphism

$$\text{Id}^{\flat} \boxtimes \Pi(\text{Id}^{\flat}) : \Pi_{\mathcal{V}_{\bullet}} \boxtimes \text{Id}_{\mathcal{W}_{\bullet}} \Longrightarrow \text{Id}_{\mathcal{V}_{\bullet}} \boxtimes \Pi_{\mathcal{W}_{\bullet}},$$

which does not exist in $\mathcal{V} \boxtimes \mathcal{W}$. We make $\mathcal{V} \boxtimes_s \mathcal{W}$ into a superlinear category with shift functor $\Pi \boxtimes \text{Id}$. It follows that the composite

$$B\boxtimes: \mathcal{V} \times \mathcal{W} \longrightarrow \mathcal{V} \boxtimes_s \mathcal{W}$$

is superlinear in both variables. The graded extension of $(\mathcal{V} \boxtimes_s \mathcal{W}, \Pi \boxtimes \text{Id})$ is naturally identified with $\mathcal{V}_{\bullet} \boxtimes \mathcal{W}_{\bullet}$. Hence (1.6.4) implies that $\boxtimes_s := B\boxtimes$ is the universal such *bi-superlinear* functor out of $\mathcal{V} \times \mathcal{W}$. More precisely, the pair $(\mathcal{V} \boxtimes_s \mathcal{W}, \boxtimes_s)$ satisfies the universal property (1.2.4) with (bi-)superlinear functors and supernatural transformations in the place of (bi-)linear functors and linear natural transformations.

(1.6.6) The abelian case. Let \mathcal{A} and \mathcal{B} be superlinear abelian categories. Then we may apply the main Theorem of [Gre10] (with $\mathcal{C} = \mathbf{SVect}^f$) to obtain the ‘*categorified coequalizer*’ in the diagram

$$\mathcal{A} \hat{\boxtimes} \mathcal{B} \begin{array}{c} \xrightarrow{\Pi \hat{\boxtimes} \text{id}} \\ \xrightarrow{\text{id} \hat{\boxtimes} \Pi} \end{array} \mathcal{A} \hat{\boxtimes} \mathcal{B} \xrightarrow{B} \mathcal{A} \hat{\boxtimes}_s \mathcal{B}$$

see [Gre10, Rem.3.6]. More precisely, a functor F out of $\mathcal{A} \hat{\boxtimes} \mathcal{B}$ is called *SVect-balanced* if there is a functor isomorphism

$$\beta: F \circ (\Pi \boxtimes \text{Id}) \implies F \circ (\text{Id} \boxtimes \Pi).$$

(This β is part of the data of balanced functor, but suppressed from the notation.) The functor B above is the universal right-exact SVect-balanced functor out of $\mathcal{A} \hat{\boxtimes} \mathcal{B}$. As in the previous paragraph, we set $\boxtimes_s := B\boxtimes$, where $\boxtimes: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \hat{\boxtimes} \mathcal{B}$ is the universal bi-right-exact, bilinear functor in (1.3.1). Since *bi-superlinear* implies *SVect-balanced*, it follows that the pair $(\mathcal{A} \hat{\boxtimes}_s \mathcal{B}, \boxtimes_s)$ satisfies the universal property in (1.3.1) with (bi-)superlinear right-exact functors and supernatural transformations in the place of (bi-)linear right-exact functors and linear natural transformations.

(1.6.7) Examples. (a) Let A and B be two finitely generated superalgebras. The proof of [Del90, Prop.5.3] can be formulated in the super setting (careful with Koszul signs!) to show that

$$\begin{aligned} (\text{Smod}^f\text{-}A) \times (\text{Smod}^f\text{-}B) &\longrightarrow \text{Smod}^f\text{-}(A \otimes B) \\ (M, N) &\longmapsto M \otimes_{\mathbf{k}} N. \end{aligned}$$

makes $\text{Smod}^f\text{-}(A \otimes B)$ the categorical tensor product of $\text{Smod}^f\text{-}A$ and $\text{Smod}^f\text{-}B$.

(b) Consider the 2-vectorspaces

$$[0|1] = \text{Smod}^f\text{-}C_1 \quad \text{and} \quad [1|0] = \text{SVect}^f \simeq \text{Smod}^f\text{-}C_2.$$

By the previous example,

$$[0|1] \hat{\boxtimes}_s [0|1] \simeq [1|0].$$

(c) Let \mathcal{V} be any superlinear abelian category. Then

$$[1|0] \boxtimes_s \mathcal{V} \simeq \mathcal{V}$$

is already abelian, so we do not need to complete the tensor product. We arrive at the following general formula for completed tensor products of superlinear 2-vectorspaces:

$$[k|l] \hat{\boxtimes}_s [m|n] \simeq [km + ln | kn + lm].$$

(d) Consider the n th Clifford superalgebra C_n defined by odd generators ξ_1, \dots, ξ_n subject to the relations

$$\xi_i^2 = 1, \quad \xi_i \xi_j = -\xi_j \xi_i, \quad i \neq j.$$

Then we have

$$C_n \cong C_1^{\otimes n}.$$

We recover the well known fact

$$\text{Smod}^f\text{-}C_n \simeq (\text{Smod}^f\text{-}C_1)^{\hat{\boxtimes}_s n} \simeq \begin{cases} [1|0] & \text{if } n \text{ is even, and} \\ [0|1] & \text{if } n \text{ is odd.} \end{cases}$$

(e) Let \mathcal{V} be a superlinear abelian category, and let A be a superalgebra of finite dimension over \mathbf{k} . Then the argument in [Del90, 5.11] is reformulated in the super-setting to identify $\mathcal{V} \hat{\boxtimes}_s (\text{Smod}^f\text{-}A)$ with the category of right A -supermodules inside \mathcal{V} .

(1.6.8) The pre-triangulated case. Let C^\bullet and D^\bullet be two 2-periodic complexes over \mathbf{k} . Then their tensor product as super-vectorspaces, endowed with the differentials

$$d^{\bar{0}} = \begin{pmatrix} d^{\bar{0}} \otimes \text{id} & -\text{id} \otimes d^{\bar{1}} \\ \text{id} \otimes d^{\bar{0}} & d^{\bar{1}} \otimes \text{id} \end{pmatrix} \quad \text{and} \quad d^{\bar{1}} = \begin{pmatrix} d^{\bar{1}} \otimes \text{id} & \text{id} \otimes d^{\bar{1}} \\ -\text{id} \otimes d^{\bar{0}} & d^{\bar{0}} \otimes \text{id} \end{pmatrix},$$

is again a 2-periodic complex. Let \mathcal{V} and \mathcal{W} be 2-periodic perfect dg-categories, and view $\mathcal{V} \boxtimes \mathcal{W}$ as 2-periodic dg-category with the differentials just defined and composition with a sign as in (1.4.9). We define the completed supertensor product of \mathcal{V} and \mathcal{W} as

$$\mathcal{V} \widehat{\boxtimes}_s \mathcal{W} := \text{Perf}^{(2)}(\mathcal{V} \boxtimes \mathcal{W}).$$

(1.7) Grothendieck groups. If \mathcal{V} is an abelian category the Grothendieck group $K(\mathcal{V})$ is defined, as usual, by generators $\langle V \rangle$, $V \in \text{Ob}(\mathcal{V})$ subject to the relations $\langle V' \rangle + \langle V'' \rangle = \langle V \rangle$ for each exact sequence

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0.$$

If \mathcal{V} is an abelian superlinear category, then we define $K(\mathcal{V}) = K^0(\mathcal{V})$ in a similar way, but imposing the additional relations

$$(1.7.1) \quad \langle \Pi V \rangle = -\langle V \rangle, \quad V \in \text{Ob}(\mathcal{V}).$$

This differs from the conventions in [Joz89] and [Kle05, (12.18)ff] by a sign. For $n \geq 1$, we let

$$K^n(\mathcal{V}) := K^0(\mathcal{V} \widehat{\boxtimes}_s \text{Smod}^f\text{-}C_n),$$

where C_n is the n th Clifford superalgebra. By (1.6.7)(d), this defines an even periodic theory. We denote by

$$(1.7.2) \quad K_{\mathbb{C}}^\bullet(\mathcal{W}) = K^\bullet(\mathcal{W}) \otimes \mathbb{C}$$

the complexified Grothendieck groups. Assuming that there exists an $i \in \mathbf{k}$ with $i^2 = -1$, we may define a product map

$$(1.7.3) \quad \otimes: K^\bullet(\mathcal{V}) \otimes K^\bullet(\mathcal{W}) \longrightarrow K^\bullet(\mathcal{V} \widehat{\boxtimes}_s \mathcal{W})$$

as in [Kle05, (12.21)], as follows: if $\deg \langle V \rangle \cdot \deg \langle W \rangle = 0$, we set

$$\langle V \rangle \otimes \langle W \rangle := \langle V \boxtimes_s W \rangle.$$

If $\deg \langle V \rangle = \deg \langle W \rangle = 1$, we use (1.6.7) to view $V \boxtimes_s W$ as a right C_2 -supermodule object inside $\mathcal{V} \widehat{\boxtimes}_s \mathcal{W}$. It follows that we have the idempotent endomorphism

$$f := \frac{1 + i\xi_1\xi_2}{2}$$

of $V \boxtimes_s W$ with

$$\ker(f)\xi_1 = \text{im}(f),$$

and we set

$$\langle V \rangle \otimes \langle W \rangle := \langle \text{im}(f) \rangle.$$

(1.7.4) Examples. (a) Let A be an associative superalgebra over \mathbf{k} , which is coherent on the right. Then the category $\text{Smod}^f\text{-}A$ of finitely presented right A -supermodules is a superlinear abelian category. On the other hand, $A = A_{\bar{0}} \oplus A_{\bar{1}}$ can be considered as an ordinary associative algebra. Assuming that this algebra is coherent on the right as well,

we have the \mathbf{k} -linear abelian category $\text{mod}^f\text{-}A$ of finitely presented right A -modules. The Grothendieck groups of $\text{Smod}^f\text{-}A$ and $\text{mod}^f\text{-}A$ may be different.

(b) An instructive example is provided by the Clifford superalgebras themselves. Considered as just an associative algebra, C_n gives

$$K(\text{mod}^f\text{-}C_n) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even, and} \\ \mathbb{Z}^2 & \text{if } n \text{ is odd.} \end{cases}$$

as C_n is either simple or has two simple summands, depending on the parity of n . However, considered as a superalgebra, C_n is simple, see [Kle05, Exa.12.1.3], and we have

$$K^\bullet(\text{Smod}^f\text{-}C_n) = \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2) & \text{if } n \text{ is even, and} \\ (\mathbb{Z}/2) \oplus \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

So, the complexified K -groups $K_{\mathbb{C}}^\bullet(\text{Smod}^f\text{-}C_n)$ are one-dimensional supervectorspaces over \mathbb{C} . Simplifications of this kind will be crucial later in the paper.

(c) Let A and B be finite dimensional superalgebras. Then

$$\otimes: K_{\mathbb{C}}^\bullet(\text{Smod}^f\text{-}A) \otimes K_{\mathbb{C}}^\bullet(\text{Smod}^f\text{-}B) \longrightarrow K_{\mathbb{C}}^\bullet(\text{Smod}^f\text{-}(A \otimes B))$$

is an isomorphism (compare [Kle05, Lemma 12.2.15]).

If \mathcal{D} is a triangulated \mathbf{k} -linear category, then one defines $K(\mathcal{D})$ by generators $\langle V \rangle$, $V \in \text{Ob}(\mathcal{V})$, and relations $\langle V' \rangle + \langle V'' \rangle = \langle V \rangle$ for each exact triangle

$$V' \rightarrow V \rightarrow V'' \rightarrow V'[1].$$

Note that this implies automatically that

$$(1.7.5) \quad \langle V[1] \rangle = -\langle V \rangle.$$

If \mathcal{V} is pre-triangulated, then we define

$$K(\mathcal{V}) := K(H^0(\mathcal{V})).$$

Note that if \mathcal{V} is a 2-periodic pre-triangulated category, then it is super-linear, with $\Pi V = V[1]$, so (1.7.4) can be written as (1.7.1).

2. SYMMETRIC POWERS.

(2.1) Reminder on 2-representations. We keep working over the field \mathbf{k} , as in (1.1). Let G be a group. A 2-representation (or categorical representation) of G is an action ϱ of G on a linear category \mathcal{V} . Explicitly, this means that we have the following data, cf. [GK07]:

(2.1.1)(a) For each element $g \in G$, a linear functor $\varrho(g) : \mathcal{V} \rightarrow \mathcal{V}$.

(2.1.1)(b) For any pair of elements (g, h) of G an isomorphism of functors

$$\phi_{g,h} : (\varrho(g) \circ \varrho(h)) \xrightarrow{\cong} \varrho(gh)$$

(2.1.1)(c) An isomorphism of functors

$$\phi_1 : \varrho(1) \xrightarrow{\cong} \text{Id}_c$$

such that the following conditions hold:

(2.1.1)(d) For any $g, h, k \in G$ we have

$$\phi_{(gh,k)}(\phi_{g,h} \circ \varrho(k)) = \phi_{(g,hk)}(\varrho(g) \circ \phi_{h,k})$$

(associativity); we also write $\phi_{g,h,k}$,

(2.1.1)(e) We have

$$\phi_{1,g} = \phi_1 \circ \varrho(g) \quad \text{and} \quad \phi_{g,1} = \varrho(g) \circ \phi_1.$$

(2.1.2) Examples. (a) Let

$$c : G \times G \rightarrow \mathbf{k}^*$$

be a 2-cocycle, i.e., a function satisfying the identity

$$c(gh, k) \cdot c(g, h) = c(g, hk) \cdot c(h, k).$$

We then have an action $\varrho = \varrho_c$ of G on Vect^f . By definition, for $g \in G$ the functor $\varrho(g) : \text{Vect}^f \rightarrow \text{Vect}^f$ is the identity, and the transformation $\phi_{g,h} : \text{Id} \Rightarrow \text{Id}$ is multiplication with $c(g, h)$, while ϕ_1 is the multiplication by $c(1, 1)$.

The 2-representation ϱ_c is equivalent to $\varrho_{c'}$ (in the appropriate sense, see [Bar11, 2.2]) if and only if c is cohomologous to c' , and $H^2(G, \mathbf{k}^*)$ is identified with the set of “1-dimensional 2-representations of G ”, i.e., G -actions on Vect^f modulo equivalence.

(b) Alternatively, let

$$1 \longrightarrow \mathbf{k}^* \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension of G . For every $g \in G$ the set $\pi^{-1}(g)$ is a \mathbf{k}^* -torsor, and therefore

$$L_g := \pi^{-1}(g) \cup \{0\}$$

is a 1-dimensional \mathbf{k} -vector space. Note that $L_1 = \mathbf{k}$. The group structure on \tilde{G} induces isomorphisms

$$\mu_{g,h} : L_g \otimes_k L_h \longrightarrow L_{gh}.$$

satisfying the associativity conditions. For each $g \in G$ we then have the functor of tensor multiplication with L_g :

$$\begin{aligned} \varrho(g) : \text{Vect}^f &\longrightarrow \text{Vect}^f \\ V &\longmapsto L_g \otimes V. \end{aligned}$$

The isomorphisms $\mu_{g,h}$ give then the isomorphisms $\phi_{g,h}$ from (2.1.1)(b), and we define the functor ϕ_1 to be the identity. This gives an action of G on Vect^f . One sees easily that if \tilde{G} corresponds to a cocycle c , then this 2-representation of G is equivalent to ϱ_c from (a).

(c) A superlinear category is the same as a \mathbf{k} -linear category with a strict linear action by $\mathbb{Z}/2$. “Strict” means that all the $\phi_{g,h}$ and ϕ_1 are identity maps. The definitions of superlinear functor and supernatural transformation in (1.6) translate into the definition of 1- and 2-morphisms of 2-representations [Bar11, 2.2.2].

(2.1.3) Given a 2-representation of G on \mathcal{V} , we denote by \mathcal{V}^G the *category of G -equivariant objects in \mathcal{V}* , i.e., objects $V \in \mathcal{V}$ equipped with isomorphisms

$$(2.1.4) \quad \epsilon_g : V \rightarrow \varrho(g)(V), \quad g \in G,$$

satisfying the following compatibility condition: for any $g, h \in G$ the diagram

$$\begin{array}{ccc} V & \xrightarrow{\epsilon_g} & \varrho(g)(V) \\ \epsilon_{gh} \downarrow & & \downarrow \varrho(g)(\epsilon_h) \\ \varrho(gh)(V) & \xleftarrow{\phi_{g,h,V}} & \varrho(g)(\varrho(h)(V)) \end{array}$$

is commutative.

(2.1.5) Example. Suppose that the G -action on \mathcal{V} is trivial: all $\varrho(g)$ and $\phi_{g,h}$, as well as ϕ_1 , are the identities. Then a G -equivariant object in \mathcal{V} is the same as a representation of G in \mathcal{V} , i.e., an object $V \in \mathcal{V}$ and a homomorphism $G \rightarrow \text{Aut}_{\mathcal{V}}(V)$.

(2.1.6) If \mathcal{V}, \mathcal{W} are two 2-representations of G , then $\mathcal{V} \boxtimes \mathcal{W}$ is again a 2-representation, in an obvious way. Further, if \mathcal{V}, \mathcal{W} are abelian and $\mathcal{V} \hat{\boxtimes} \mathcal{W}$ exists, then it is also a 2-representation of G . This follows from the characterization of $\mathcal{V} \hat{\boxtimes} \mathcal{W}$ by a (2-)universal property. Similarly, if \mathcal{V}, \mathcal{W} are perfect dg-categories with G -action, then so is $\mathcal{V} \hat{\boxtimes} \mathcal{W}$.

(2.1.7) Example. For 1-dimensional 2-representations \mathcal{V}_c as in Example 2.1.2(a), we have

$$\mathcal{V}_c \boxtimes \mathcal{V}_{c'} \simeq \mathcal{V}_{c \cdot c'}.$$

So $H^2(G, \mathbf{k}^*)$ is interpreted as the Picard group of 1-dimensional 2-representations, with operation given by \boxtimes .

(2.2) Symmetric powers. Let \mathcal{V} be a linear category. We have then the linear category

$$\mathcal{V}^{\boxtimes n} = \mathcal{V} \boxtimes \dots \boxtimes \mathcal{V} \quad (n \text{ times}).$$

Let S_n denote the symmetric group of permutations of $\{1, \dots, n\}$. Then we have the S_n -action on $\mathcal{V}^{\boxtimes n}$ given by

$$\sigma(V_1 \boxtimes \dots \boxtimes V_n) = V_{\sigma^{-1}(1)} \boxtimes \dots \boxtimes V_{\sigma^{-1}(n)},$$

and extended to direct sums by additivity.

(2.2.1) Definition. The n th symmetric power $\mathrm{Sym}^n(\mathcal{V})$ is the category of S_n -equivariant objects in $\mathcal{V}^{\boxtimes n}$:

$$\mathrm{Sym}^n(\mathcal{V}) = (\mathcal{V}^{\boxtimes n})^{S_n}.$$

(2.2.2) Example. Let \mathcal{V} be the standard 1-dimensional 2-vector space, so $\mathcal{V} = [1] = \mathrm{Vect}^f$, see (1.2.2). Then $\mathcal{V}^{\boxtimes n} = \mathrm{Vect}^f$ and the S_n -action is trivial. So by Example 2.1.5, we have that

$$\mathrm{Sym}^n(\mathcal{V}) = \mathrm{Rep}(S_n)$$

is the category of finite-dimensional representations of S_n over \mathbf{k} . Assume $\mathrm{char}(\mathbf{k}) = 0$. Denote by $p(n)$ the number of partitions of n . Then, as well known, $\mathrm{Rep}(S_n)$ has $p(n)$ simple objects (irreducible representations of S_n). So we can write

$$(2.2.3) \quad \mathrm{Sym}^n[1] \simeq [p(n)].$$

This formula indicates that tensor operations on categories lead to a new type of λ -rings, different from the standard (special) λ -rings which, as well known, satisfy

$$\mathrm{Sym}^n(1) = 1, \quad \forall n.$$

We will study such “2-special” λ -rings in a separate paper.

(2.2.4) Proposition. We have an equivalence

$$\mathrm{Sym}^n(\mathcal{V} \boxplus \mathcal{W}) \simeq \bigoplus_{i+j=n} \mathrm{Sym}^i(\mathcal{V}) \boxtimes \mathrm{Sym}^j(\mathcal{W}).$$

Proof: This follows from Proposition 1.1.6. □

Let us denote by

$$(2.2.5) \quad \phi(q) = \left(\sum_{n \geq 0} p(n) q^n \right)^{-1} = \prod_{n \geq 1} (1 - q^n)$$

the Euler function.

(2.2.6) Corollary. Assume $\mathrm{char}(\mathbf{k}) = 0$. Let \mathcal{V} be a 2-vector space of dimension d . Then

$$\sum_{n \geq 0} \mathrm{Dim} \, \mathrm{Sym}^n(\mathcal{V}) q^n = \frac{1}{\phi(q)^d}.$$

(2.2.7) If \mathcal{V} is abelian or perfect, then we can form the completed symmetric product $\widehat{\mathrm{Sym}}^n(\mathcal{V})$. For this, we start with the completed tensor power

$$\mathcal{V}^{\widehat{\boxtimes} n} = \mathcal{V}^{\widehat{\boxtimes}} \cdots \widehat{\boxtimes} \mathcal{V},$$

which we assume to exist in the abelian case, and define as in (1.4.9) in the perfect case. Then $\widehat{\mathrm{Sym}}^n(\mathcal{V})$ is defined as the category of S_n -equivariant objects in $\mathcal{V}^{\widehat{\boxtimes} n}$.

(2.2.8) Examples.(a) Let X be a smooth projective variety. Then $\widehat{\mathrm{Sym}}^n(I(X))$ is the dg-category of S_n -equivariant complexes of injective \mathcal{O} -modules on X^n that are bounded below and have bounded coherent cohomology. The corresponding H^0 -category is the bounded derived category of S_n -equivariant complexes of coherent sheaves on X^n .

(b) Let A be a finite-dimensional \mathbf{k} -algebra, and $\mathcal{V} = \text{mod}^f\text{-}A$. Then $V^{\widehat{\otimes}^n}$ is the category of right $A^{\otimes n}$ -modules. The group S_n acts on $A^{\otimes n}$ by algebra automorphisms, so we have the crossed product algebra $A^{\otimes n}[S_n]$, and

$$\widehat{Sym}^n(\mathcal{V}) = \text{mod}^f\text{-}A^{\otimes n}[S_n].$$

3. EXTERIOR POWERS.

(3.1) Picard categories and categorical characters. The construction of exterior powers in usual linear algebra is based on two steps:

- (1) Identification of the sign character of S_n as a homomorphism $\text{sgn} : S_n \rightarrow \mathbb{Z}/2$ which is universal among homomorphisms $S_n \rightarrow A$ into abelian groups. At this stage $\mathbb{Z}/2$ is considered as an abstract 2-element group.
- (2) Realization of $\mathbb{Z}/2$ as the subgroup $\{\pm 1\} \subset \mathbf{k}^*$ of the multiplicative group of the field, which is then used to multiply elements of various vector spaces.

We start by discussing the categorical analog of (1). For this, abelian groups A should be replaced by Picard categories. We recall basic definitions and examples, following [SGA4] Exp. XVIII.

(3.1.1) Definition. By a (*symmetric*) *Picard category* we mean a (symmetric) monoidal category $(\mathcal{A}, \otimes, \mathbf{1})$ in which all the objects are invertible, and all the morphisms are invertible with respect to composition.

For any essentially small Picard category \mathcal{A} we have the groups

$$\pi_0(\mathcal{A}) = (\text{Ob}(\mathcal{A})/\text{iso}, \otimes), \quad \pi_1(\mathcal{A}) = \text{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}).$$

The group $\pi_1(\mathcal{A})$ is always abelian. If \mathcal{A} is symmetric, then $\pi_0(\mathcal{A})$ is also abelian.

(3.1.2) Examples. (a) Let G be a group. We can consider the set G as a discrete category (the set of objects is G , the only morphisms are identities). The group operation in G makes then G into a Picard category, which we will still denote G and call the *discrete Picard category* associated to G . This Picard category has $\pi_0 = G$, $\pi_1 = 0$. If G is abelian, then we get a symmetric Picard category.

(b) Let A be an abelian group. Denote by $A\text{-}\mathcal{Tors}$ the category of A -torsors (i.e., principal homogeneous spaces T over A) and their isomorphisms. This category is a symmetric Picard category with respect to the tensor product of torsors defined by

$$T \otimes_A T' = (T \times T') / \{(at, t') \sim (t, at'), \quad a \in A, t \in T, t' \in T'\}.$$

Note that $A\text{-}\mathcal{Tors}$ is equivalent to the Picard category $\mathcal{B}A$ with one object $\mathbf{1}$ and $\text{Hom}(\mathbf{1}, \mathbf{1}) = A$. The operation \otimes on morphisms, as well as the composition of morphisms in $\mathcal{B}A$, are both given by the group operation in A . Both $A\text{-}\mathcal{Tors}$ and $\mathcal{B}A$ have $\pi_0 = 0$, $\pi_1 = A$.

(c) Let $A^\bullet = \{A^0 \xrightarrow{d} A^1\}$ be a two-term complex of abelian groups. Then we have a symmetric Picard category $[A^\bullet]$ with

$$\text{Ob}([A^\bullet]) = A^1, \quad \text{Hom}_{[A^\bullet]}(a, a') = \{b \in A^0 : d(b) = a - a'\}.$$

Note that

$$\pi_i[A^\bullet] = H^{1-i}(A^\bullet), \quad i = 0, 1.$$

(d) Let \mathbf{k} be a field. We have then the symmetric Picard category $\mathcal{P}ic^{\mathbb{Z}}(\mathbf{k})$ of *graded lines* over \mathbf{k} . By definition, objects of $\mathcal{P}ic^{\mathbb{Z}}(\mathbf{k})$ are \mathbb{Z} -graded \mathbf{k} -vector spaces $V = \bigoplus_{n \in \mathbb{Z}} V_n$ of total dimension 1. In other words, such a V has exactly one graded component of dimension 1, while all other graded components vanish. Morphisms in $\mathcal{P}ic^{\mathbb{Z}}(\mathbf{k})$ are isomorphisms of

graded vector spaces. The monoidal operation is given by the usual graded tensor product, while the symmetry is given by the Koszul sign rule:

$$x \otimes y \longmapsto (-1)^{\deg(x)\deg(y)} y \otimes x.$$

We also denote by $\mathcal{Pic}^{\mathbb{Z}/2}(\mathbf{k})$ the symmetric Picard category of *super-lines* over \mathbf{k} by which we mean super-vector spaces over \mathbf{k} of total dimension 1, i.e., of super-dimension either $(1|0)$ or $(0|1)$. We have

$$\pi_0(\mathcal{Pic}^{\mathbb{Z}}(\mathbf{k})) = \mathbb{Z}, \quad \pi_1(\mathcal{Pic}^{\mathbb{Z}}(\mathbf{k})) = \mathbf{k}^*, \quad \pi_0(\mathcal{Pic}^{\mathbb{Z}/2}(\mathbf{k})) = \mathbb{Z}/2, \quad \pi_1(\mathcal{Pic}^{\mathbb{Z}/2}(\mathbf{k})) = \mathbf{k}^*.$$

(3.1.3) Definition. Let G be a group and \mathcal{A} be a symmetric Picard category. An \mathcal{A} -valued *Picard character* of G is a monoidal functor $X : G \rightarrow \mathcal{A}$ (where G is considered as a discrete Picard category).

Explicitly, a Picard character consists of objects $X(g) \in \mathcal{A}$ given for each $g \in G$, of isomorphisms

$$\phi_{g,h} : X(g) \otimes X(h) \rightarrow X(gh),$$

and of an isomorphism $\phi_1 : X(1) \rightarrow \mathbf{1}$ which satisfy the conditions identical to (2.1.1)(d-e), with \circ replaced by \otimes .

In fact, both Picard characters and 2-representations as defined in (2.1), are particular cases of a more general concept: a 2-representation of G in a 2-category \mathcal{C} , see [GK07] Def. 4.1. More precisely, the context of (2.1) corresponds to \mathcal{C} being the 2-category of \mathbf{k} -linear categories, their linear functors and natural transformations. Definition (3.1.3) corresponds to the case when \mathcal{C} is the 2-category with one object pt canonically associated to the monoidal category \mathcal{A} : here

$$1\text{Hom}_{\mathcal{C}}(\text{pt}, \text{pt}) = \mathcal{A}$$

and composition of 1-morphisms corresponds to the monoidal structure on \mathcal{A} .

(3.1.4) Examples. (a) If $\mathcal{A} = A$ is a discrete symmetric Picard category, then an A -valued Picard character is simply a group homomorphism $G \rightarrow A$. More generally, for any symmetric Picard category \mathcal{A} and any Picard character $G \rightarrow \mathcal{A}$ we obtain a homomorphism $G \rightarrow \pi_0(\mathcal{A})$ with an abelian group target.

(b) If $\mathcal{A} = A - \mathcal{Tors}$ is the category of torsors over an abelian group A , then an \mathcal{A} -valued Picard character of G is the same as a central extension

$$1 \rightarrow A \longrightarrow \tilde{G} \xrightarrow{p} G \rightarrow 1.$$

Indeed, given such an extension, every preimage

$$X(g) = p^{-1}(g)$$

is a torsor over $A = p^{-1}(1)$, and the group operation in \tilde{G} gives identifications

$$X(g) \otimes_A X(g') \longrightarrow X(gg').$$

Conversely, given any Picard character $X : G \rightarrow \mathcal{A} - \mathcal{Tors}$, the set

$$\tilde{G} = \coprod_{g \in G} X(g)$$

becomes a group, fitting into a central extension as above. This generalizes Example 2.1.2(b) which corresponds to $A = \mathbf{k}^*$.

(c) If $\mathcal{A} = \mathcal{B}A$ for an abelian group A , then an \mathcal{A} -valued Picard character is the same as a 2-cocycle $c : G \times G \rightarrow A$.

(3.1.5) Symmetric Picard categories and $[0, 1]$ -spectra. The construction of symmetric Picard categories in Example(3.1.2)(c) is not the most general one. It is an insight of Grothendieck (see [Dr06] (5.5.2) for a discussion) that a complete description can be obtained in terms of spectra, i.e., objects of the stable homotopy category.³ We recall some details of this description. A *spectrum* Y can be seen as an infinite loop space, i.e., a topological space $\Omega^\infty Y$ together with a sequence of its deloopings $\Omega^{\infty-j} Y$, $j \geq 0$, which are topological spaces equipped with homotopy equivalences $\Omega^j(\Omega^{\infty-j} Y) \sim \Omega^\infty Y$. In particular, a spectrum Y has homotopy groups $\pi_i(Y)$, $i \in \mathbb{Z}$, which are defined as

$$\pi_i(Y) = \pi_{i+j}(\Omega^{\infty-j}(Y)), \quad j \gg 0.$$

By a $[0, 1]$ -*spectrum* we mean a spectrum Y which has only two nonzero homotopy groups, namely $\pi_0(Y)$ and $\pi_1(Y)$.

Note that any Picard category \mathcal{A} , considered as an abstract category, is a groupoid. So for a small \mathcal{A} the classifying space $B(\mathcal{A})$ is a topological space with

$$\pi_0(B(\mathcal{A})) = \pi_0(\mathcal{A}), \quad \pi_1(B(\mathcal{A}), \{1\}) = \pi_1(\mathcal{A}),$$

while the higher homotopy groups of each component of $B(\mathcal{A})$ vanish.

(3.1.6) Theorem. (a) The functor of taking the classifying space extends to a functor

$$\mathbb{B} : \{\text{Small Symmetric Picard Categories}\} \longrightarrow \{[0, 1]\text{-spectra}\}$$

such that $\pi_i(\mathbb{B}(\mathcal{A})) = \pi_i(\mathcal{A})$, $i = 0, 1$.

(b) The functor \mathbb{B} takes equivalences of symmetric Picard categories into homotopy equivalence of $[0, 1]$ -spectra. After inverting the two types of equivalences, B^∞ becomes an equivalence of localized categories.

For convenience of the reader we sketch the construction of the functor \mathbb{B} . The infinite loop space of the spectrum $\mathbb{B}(\mathcal{A})$ is, by definition, $\mathbb{B}^\infty(\mathcal{A}) = B(\mathcal{A})$, the usual classifying space. For any $m \geq 0$ the m th delooping $\mathbb{B}^{\infty-m}(\mathcal{A})$ is defined as the geometric realization of the simplicial set $\mathbb{B}_\bullet^{\infty-m}(\mathcal{A})$ constructed as follows. By definition, n -simplices of $\mathbb{B}_\bullet^{\infty-m}(\mathcal{A})$ are “ \mathcal{A} -valued m -cocycles on the standard simplex $\Delta[n]$ ”, i.e., data consisting of:

- An object $X_\rho \in \mathcal{A}$ for any m -dimensional face $\rho \subset \Delta^m$.
- An (iso)morphism

$$\phi_\sigma : \bigotimes_{i \equiv 0 \pmod{2}} X_{\partial_i \sigma} \longrightarrow \bigotimes_{i \equiv 1 \pmod{2}} X_{\partial_i \sigma}$$

for each $(m+1)$ -dimensional face $\sigma \subset \Delta[n]$.

- For any $(m+2)$ -dimensional face $\tau \subset \Delta[n]$ the isomorphisms $\phi_{\partial_i \tau}$ must satisfy a compatibility condition, identical to one spelled out in [Pre], §2.9 (for the case when \mathcal{A} is the category of torsors over an abelian group A).

Note that for $m = 0$ this gives the usual definition of the nerve of \mathcal{A} . Further details are left to the reader. \square

³For an introduction to spectra, see [Ada78].

(3.1.7) Examples. (a) The Picard category $[A^\bullet]$ from Example (3.1.2)(c) corresponds to the Eilenberg-MacLane spectrum $EM(V^\bullet)$.

(b) Let Y be any *connective* spectrum, i.e., a spectrum such that $\pi_i(Y) = 0$ for $i < 0$. Then we have a well defined truncation $Y_{[0,1]}$ of Y in the homotopy degrees 0, 1, which is a $[0, 1]$ -spectrum and can therefore be described in terms of an appropriate Picard category. We consider several particular cases.

(c) (K-theory spectra.) Let \mathcal{E} be an exact category in the sense of Quillen and $\mathcal{K}(\mathcal{E})$ be its algebraic K-theory spectrum, so $\pi_i \mathcal{K}(\mathcal{E}) = K_i(\mathcal{E})$ are the algebraic K-groups of \mathcal{E} . The symmetric Picard category corresponding to the $[0, 1]$ -spectrum $\mathcal{K}(\mathcal{E})_{[0,1]}$ was described by Deligne in [Del87]. It was called the *category of virtual objects* of \mathcal{E} and denoted $\mathcal{V}(\mathcal{E})$. This Picard category can be characterized as the target of the universal determinantal theory on \mathcal{E} , see [Pre], Ex. 2.13, 2.29.

(d) (K-theory of a field.) Let \mathbf{k} be a field and \mathcal{E} be the category of finite-dimensional \mathbf{k} -vector spaces. In this case

$$K_0(\mathcal{E}) = K_0(\mathbf{k}) = \mathbb{Z}, \quad K_1(\mathcal{E}) = K_1(\mathbf{k}) = \mathbf{k}^\times.$$

It was observed in [Del87] and emphasized in [Dr06] that $\mathcal{V}(\mathcal{E})$ is equivalent, as a symmetric Picard category, to the category $\mathcal{Pic}^{\mathbb{Z}}(\mathbf{k})$ from Example 3.1.2(d).

(e) (The spherical spectrum.) Let \mathbb{S} be the spherical spectrum, so $\pi_i(\mathbb{S}) = \pi_i^{\text{st}}$ are the stable homotopy groups of spheres:

$$\pi_0^{\text{st}} = \mathbb{Z}, \quad \pi_1^{\text{st}} = \mathbb{Z}/2, \quad \pi_2^{\text{st}} = \mathbb{Z}/2, \quad \pi_3^{\text{st}} = \mathbb{Z}/24, \text{ etc.}$$

As \mathbb{S} is connective, we have the truncation $\mathbb{S}_{[0,1]}$. The corresponding Picard category can be viewed as a free symmetric Picard category generated by one object. It can be described as a modification of the category $\mathcal{Pic}^{\mathbb{Z}}(\mathbf{k})$. More precisely, let $\mathcal{Pic}^{\mathbb{Z}}(\mathbb{Z})$ be the category of \mathbb{Z} -graded abelian groups $L = \bigoplus_{n \in \mathbb{Z}} L_n$ which are free of total rank 1. Morphisms are isomorphisms of graded abelian groups. Tensor product and symmetry are defined similarly to $\mathcal{Pic}^{\mathbb{Z}}(\mathbf{k})$.

In other words, the Picard category classifying (the truncation of) the spherical spectrum \mathbb{S} can be seen as the “sign skeleton” of the category of graded vector spaces with Koszul sign rule: it contains exactly the minimal data necessary to write down this rule. So it is the structure of \mathbb{S} which ultimately leads to the sign rules of super-mathematics.

(f) (The loop space of the spherical spectrum.) Remarkably, the spherical spectrum leads to super-constructions not in one, but in two ways. More precisely, let \mathbb{S}_0 be the connected component of \mathbb{S} corresponding to the zero element of $\pi_0(\mathbb{S}) = \mathbb{Z}$. The loop space (spectrum) $\Omega \mathbb{S}_0$ is then a connective spectrum whose homotopy groups are the π_i^{st} but with shifted numeration and with $\pi_0^{\text{st}} = \mathbb{Z}$ disregarded:

$$\pi_0(\Omega \mathbb{S}_0) = \mathbb{Z}/2, \quad \pi_1(\Omega \mathbb{S}_0) = \mathbb{Z}/2, \quad \pi_2(\Omega \mathbb{S}_0) = \mathbb{Z}/24, \text{ etc.}$$

Therefore the truncation $(\Omega \mathbb{S}_0)_{[0,1]}$ gives rise to a symmetric Picard category \mathcal{P} with

$$\pi_0(\mathcal{P}) = \pi_1(\mathcal{P}) = \mathbb{Z}/2.$$

(3.1.8) Theorem. As a symmetric Picard category, \mathcal{P} is equivalent to the category $\mathcal{Pic}^{\mathbb{Z}/2}(\mathbb{Z})$, whose objects are $\mathbb{Z}/2$ -graded free abelian groups of total rank 1, with the usual $\mathbb{Z}/2$ -graded tensor product and the Koszul sign rule for the symmetry.

In other words, the truncated spectrum $(\Omega\mathbb{S}_0)_{[0,1]}$ can be identified with the “reduction modulo 2” of $\mathbb{S}_{[0,1]}$.

We will not prove Theorem (3.1.8) in this paper but will use it as a topological motivation for our explicit constructions involving symmetric groups. Let us explain the connection in more detail.

(3.1.9) The higher sign character of S_n via the Barratt-Priddy-Quillen theorem. The classical Barratt-Priddy-Quillen (BPQ) theorem [Pri70, BP72] gives a construction of the spherical spectrum \mathbb{S} in terms of the symmetric groups S_n , $n \geq 1$. More precisely, let $S_\infty = \varinjlim_n S_n$ be the union of the standard chain of embeddings and BS_∞ be its classifying space. The alternating group $A_n \subset S_n$ is, for $n \geq 5$, equal to its commutant, and the same is true for the union $A_\infty \subset S_\infty$. In this situation, one can form the *Quillen plus-construction* $BS_\infty \rightarrow B^+S_\infty$. It can be characterized uniquely (up to homotopy equivalence) as the map reducing $\pi_1(BS_\infty) = S_\infty$ to $S_\infty/A_\infty = \mathbb{Z}/2$ and inducing an isomorphism on integral homology groups. The BPQ theorem says that $B^+S_\infty = \Omega^\infty\mathbb{S}_0$ is the connected component of the infinite loop space corresponding to the spherical spectrum. Further, the spectrum structure (i.e., the deloopings of B^+S_∞) can be canonically recovered using the semigroup structure on B^+S_∞ coming from the direct sum embeddings $S_m \times S_n \rightarrow S_{m+n}$.

The BPQ theorem implies that we have a canonical map

$$BS_n \longrightarrow BS_\infty \longrightarrow B^+S_\infty \sim \Omega^\infty\mathbb{S}_0.$$

Passing to the loop spaces, we get a map

$$\mathbf{sgn} : S_n \longrightarrow \Omega^\infty(\Omega\mathbb{S}_0)$$

which we want to call the *homotopy-theoretic sign character*. This map and its (higher) categorical manifestations should be the right tool for studying analogs of exterior powers in the context not only of categories, as in this paper, but of higher categories as well.

In particular, replacing the spectrum $\Omega\mathbb{S}_0$ by its $[0,1]$ -truncation and then by the Picard category \mathcal{P} as in Example (3.1.7)(f), we get a canonically defined Picard character

$$\mathrm{Sgn} : S_n \longrightarrow \mathcal{P},$$

which we call the *categorical sign character* of S_n . On the level of π_0 , it induces the ordinary sign character sgn .

Combining the above with Theorem (3.1.8), we get a *canonically defined* Picard character of S_n with values in the Picard category $\mathcal{Pic}^{\mathbb{Z}/2}(\mathbb{Z})$ of super-lines over \mathbb{Z} . As $\mathcal{Pic}^{\mathbb{Z}/2}(\mathbb{Z})$ naturally acts on any super-linear category, we get a conceptual explanation of the following well known phenomenon: the theory of projective representations of symmetric groups becomes much streamlined and simplified by systematic use of super-objects. One can find many instances of this phenomenon in [Kle05]. In the following sections we define the Picard character $\mathrm{Sgn} : S_n \rightarrow \mathcal{Pic}^{\mathbb{Z}/2}(\mathbb{Z})$ in an elementary way and apply it to the construction of exterior powers of categories.

(3.2) The central extension of S_n . Naive exterior powers. Assume $\mathrm{char}(\mathbf{k}) \neq 2$. The sign character (1-dimensional representation) $\mathrm{sgn} : S_n \rightarrow \{\pm 1\}$ generates the group

$$H^1(S_n, \mathbf{k}^*) = \mathrm{Hom}(S_n, \mathbf{k}^*) = \mathbb{Z}/2.$$

So the most straightforward way of generalizing it to the categorical case would be to use 1-dimensional 2-representations of S_n , which correspond (Examples 2.1.2) to elements of $H^2(S_n, \mathbf{k}^*)$. It is well known, see [Kle05], that

$$H^2(S_n, \mathbf{k}^*) = \mathbb{Z}/2, \quad n \geq 4.$$

This cohomology group is generated by the class of the central extension

$$(3.2.1) \quad 1 \rightarrow \{\pm 1\} \rightarrow \tilde{S}_n \xrightarrow{\varpi} S_n \rightarrow 1,$$

which makes sense for any $n \geq 1$ and was first found by I. Schur [Sch11, p.164]. To describe \tilde{S}_n , recall that S_n is generated by elementary transpositions

$$(3.2.2) \quad \sigma_i = (i, i+1), \quad i = 1, \dots, n-1,$$

subject to the relations

$$(3.2.3) \quad \sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2.$$

Now, \tilde{S}_n is generated by elements $s_1, \dots, s_{n-1}, \zeta$ subject to the following relations:

$$(3.2.4)(a) \quad \zeta^2 = 1, \quad \zeta s_i = s_i \zeta;$$

$$(3.2.4)(b) \quad s_i^2 = \zeta, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = \zeta s_j s_i, \quad |i-j| \geq 2.$$

The central subgroup $\{\pm 1\}$ in \tilde{S}_n is generated by ζ .

The extension \tilde{S}_n of S_n by $\{\pm 1\}$ gives, after the change of groups $\{\pm 1\} \hookrightarrow \mathbf{k}^*$, an extension by \mathbf{k}^* . Let us describe this extension in the spirit of Example 2.1.2(b), i.e., in terms of 1-dimensional \mathbf{k} -vector spaces

$$(3.2.5) \quad L_\sigma = (\varpi^{-1}(\sigma) \otimes_{\{\pm 1\}} \mathbf{k}^*) \cup \{0\}, \quad \sigma \in S_n.$$

Let $D(\sigma)$ be the set of all reduced decompositions (i.e., decomposition of the minimal possible length $l(\sigma)$)

$$(3.2.6) \quad \sigma = \sigma_{i_1} \dots \sigma_{i_l}, \quad l = l(\sigma).$$

We view an element of $D(\sigma)$ as a sequence of indices

$$\mathbf{i} = (i_1, \dots, i_l), \quad i_\nu \in \{1, \dots, n-1\}.$$

Any two decompositions $\mathbf{i}, \mathbf{i}' \in D(\sigma)$ can be transformed into each other by a sequence of moves of two types: *hexagonal moves*

$$\mathbf{i} = (\dots, i, i+1, i, \dots) \rightsquigarrow (\dots, i+1, i, i+1, \dots) = \mathbf{i}',$$

and *square moves*

$$\mathbf{i} = (\dots, i, j, \dots) \rightsquigarrow (\dots, j, i, \dots) = \mathbf{i}', \quad |i-j| \geq 2.$$

These moves correspond to implementing the second and third type of relations in (3.2.3).

(3.2.7) Proposition. (a) For any sequence of moves transforming \mathbf{i} into \mathbf{i}' the number of square moves has the same parity.

(b) The line L_σ is identified with the space of functions $f : D(\sigma) \rightarrow \mathbf{k}$ which are unchanged under hexagonal moves and change sign under each square move.

Proof: Part (b) is a direct translation of the definition of \tilde{S}_n by generators and relations (3.2.4). Indeed, any $\mathbf{i} \in D(\sigma)$ gives a lift

$$(3.2.8) \quad \tilde{\sigma}_{\mathbf{i}} = s_{i_1} \dots s_{i_l} \in \tilde{S}_n, \quad \varpi(\tilde{\sigma}_{\mathbf{i}}) = \sigma,$$

and these lifts behave as claimed under the two types of moves.

Part (a) is an implicit preliminary to (b), expressing the fact that nonzero f as in (b) exist. It is equivalent to the (classical) fact that the group defined by the relations (3.2.4) does not collapse, i.e., is indeed an extension as in (3.2.1). \square .

Let us choose a reduced decomposition for each $\sigma \in S_n$. Then we get a lifting $\tilde{\sigma}$ for each σ , so we have a 2-cocycle

$$(3.2.9) \quad c : S_n \times S_n \rightarrow \{\pm 1\}, \quad c(\sigma, \tau) = (\widetilde{\sigma\tau})(\tilde{\tau})^{-1}(\tilde{\sigma})^{-1},$$

describing the extension.

(3.2.10) Definition. (a) The naive sign 2-representation $\text{Sgn}_{\text{naive}}$ of S_n is the 1-dimensional 2-representation \mathcal{V}_c , see Example 2.1.2(a), corresponding to c from (3.2.9).

(b) Equivalently, $\text{Sgn}_{\text{naive}}$ can be described as the action of S_n on Vect^f by the functors

$$\varrho(\sigma) : V \mapsto L_\sigma \otimes V, \quad \sigma \in S_n,$$

and $\phi_{\sigma, \tau}$ being induced by the multiplicativity isomorphisms $L_\sigma \otimes L_\tau \rightarrow L_{\sigma\tau}$.

The following is a categorical analog of the main construction in [Di99].

(3.2.11) Definition. Let \mathcal{V} be a linear category. The naive exterior power of \mathcal{V} is defined to be the category

$$\Lambda_{\text{naive}}^n(\mathcal{V}) = (\mathcal{V}^{\boxtimes n} \boxtimes \text{Sgn}_{\text{naive}})^{S_n}.$$

By restricting to the generators of S_n , we get the following:

(3.2.12) Reformulation. Explicitly, an object of $\Lambda_{\text{naive}}^n(\mathcal{V})$ consists of an object $V \in \mathcal{V}^{\boxtimes n}$ together with isomorphisms

$$s_i : V \rightarrow \sigma_i^*(V), \quad i = 1, \dots, n-1,$$

satisfying the following conditions:

(a) For any $i = 1, \dots, n-1$, the composition

$$V \xrightarrow{s_i} \sigma_i^*(V) \xrightarrow{\sigma_i^*(s_i)} \sigma_i^* \sigma_i^* V \xrightarrow{\sim} V$$

is equal to $(-\text{Id}_V)$.

(b) For any $i = 1, \dots, n-2$ the two compositions

$$V \xrightarrow{s_i} \sigma_i^* V \xrightarrow{\sigma_i^*(s_{i+1})} \sigma_i^* \sigma_{i+1}^* V \xrightarrow{\sigma_i^* \sigma_{i+1}^*(s_i)} \sigma_i^* \sigma_{i+1}^* \sigma_i^* V,$$

$$V \xrightarrow{s_{i+1}} \sigma_{i+1}^* V \xrightarrow{\sigma_{i+1}^*(s_i)} \sigma_{i+1}^* \sigma_i^* V \xrightarrow{\sigma_{i+1}^* \sigma_i^*(s_{i+1})} \sigma_{i+1}^* \sigma_i^* \sigma_{i+1}^* V$$

become equal after identifying the rightmost terms with

$$(\sigma_i \sigma_{i+1} \sigma_i)^* V = (\sigma_{i+1} \sigma_i \sigma_{i+1})^* V.$$

(c) For any i, j such that $|i - j| \geq 2$ the two compositions

$$V \xrightarrow{s_i} \sigma_i^* V \xrightarrow{\sigma_i^*(s_j)} \sigma_i^* \sigma_j^* V, \quad V \xrightarrow{s_j} \sigma_j^* V \xrightarrow{\sigma_j^*(s_i)} \sigma_j^* \sigma_i^* V$$

differ by sign, after identifying the rightmost terms with

$$(\sigma_i \sigma_j)^* V = (\sigma_j \sigma_i)^* V.$$

(3.3) Projective representations of S_n . Role of superalgebra. Let c be a cocycle defining \tilde{S}_n , as in (3.2.9). Denote by $\text{Rep}_c(S_n)$ the category of finite-dimensional projective representations of S_n with central charge c , i.e., of vector spaces V together with a map $\varrho : S_n \rightarrow \text{Aut}(V)$ satisfying

$$\varrho(\sigma\tau) = c(\sigma, \tau) \cdot \varrho(\sigma)\varrho(\tau).$$

More invariantly, an object of $\text{Rep}_c(S_n)$ is a representation of \tilde{S}_n with ζ acting by (-1) . By passing to the generators of \tilde{S}_n , we see that an object of $\text{Rep}_c(S_n)$ is the same as $V \in \text{Vect}^f$ together with automorphisms $s_i : V \rightarrow V$ satisfying the relations

$$(3.3.1) \quad s_i^2 = -1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = -s_j s_i, \quad |i - j| \geq 2.$$

Let $\mathbf{k}[S_n]^c$ be the associative \mathbf{k} -algebra with generators s_i subject to these relations. It is the quotient of the group algebra of \tilde{S}_n by the relation $\zeta = -1$. Then we can say that $\text{Rep}_c(S_n)$ is identified with the category of finite-dimensional $\mathbf{k}[S_n]^c$ -modules.

(3.3.2) Example. Taking $\mathcal{V} = \text{Vect}^f = [1]$, as in Example 2.2.2, we find, by comparing (3.2.12) with (3.3.1), that

$$\Lambda_{\text{naive}}^n(\mathcal{V}) = \text{Rep}_c(S_n).$$

Irreducible objects of $\text{Rep}_c(S_n)$ in characteristic 0 were described by Schur [Sch11]. We recall his results.

(3.3.3) Definition. A *strict partition* of n is a partition

$$n = \sum_{k \geq 1} k N_k,$$

such that all the coefficients N_k are either 0 or 1. The number of strict partitions of n will be denoted $s(n)$. A partition of n is called *even* (respectively *odd*) if the corresponding conjugacy class in S_n is even (respectively odd).

(3.3.4) Theorem (Schur). Assume $\text{char}(\mathbf{k}) = 0$. Then:

(a) Irreducible objects of $\text{Rep}_c(S_n)$ are labelled by data consisting of a strict partition λ of n together with, if λ is odd, a sign $+$ or $-$. Thus for an even λ there is one irreducible representation V_λ , while an odd λ gives rise to two representations V_λ^+, V_λ^- .

(b) If λ is even then

$$V_\lambda \otimes \text{sgn} = V_\lambda,$$

while for λ odd we have

$$V_\lambda^+ \otimes \text{sgn} = V_\lambda^-.$$

□

Note that the tensor product of an object of $\text{Rep}_c(S_n)$ and an actual representation of S_n is again an object of $\text{Rep}_c(S_n)$, thus giving sense to (b).

(3.3.5) Remarks. (a) If we understand the term “projective representation” in the classical sense, as a homomorphism to PGL_N for some N , then V_λ^+ and V_λ^- give the same homomorphism. So, conjugation classes of irreducible homomorphisms $S_n \rightarrow PGL_N$ not lifting to GL_N are in bijection with strict partitions of n . This is the original formulation of Schur ([Sch11], p. 156).

(b) Note also that the generating function for strict partitions

$$(3.3.6) \quad \sum_{n \geq 0} s(n)q^n = \prod_{n \geq 1} (1 + q^n)$$

reminds of $\phi(q)$, which is the inverse of the generating function for

$$p(n) = \text{Dim Sym}^n(\text{Vect}^f).$$

This is the type of relation existing between the generating functions for exterior and symmetric powers in usual linear algebra:

$$(3.3.7) \quad \left(\sum \dim \Lambda^n(V) q^n \right) \cdot \left(\sum \dim S^n(V) (-q)^n \right) = 1.$$

This suggests that one should modify the “naive” definitions of Sgn and $\Lambda^n(\mathcal{V})$ above so as to get $\Lambda^n(\text{Vect}^f)$ satisfying an analog of (3.3.7).

It turns out that the key to such a modification is provided by the superalgebra point of view [Joz89] on projective representations of S_n . It begins with the observation that the third group of relations in (3.3.1) (anti-commutativity of the s_i) can be seen as an instance of super-commutativity. For this to hold, we have to consider the twisted group algebra $\mathbf{k}[S_n]^c$ as a super-algebra, i.e. equip it with the $\mathbb{Z}/2$ -grading defined by $\deg(s_i) = \bar{1} \in \mathbb{Z}/2$. We will now develop some additional features of this approach.

(3.4) Super 2-representations

(3.4.1) Definition. By a super 2-representation of a group G , we will mean an action of G on a superlinear category \mathcal{V} by superlinear functors and supernatural transformations.

(3.4.2) Example. By (1.6.3), a superlinear action of G on SVect^f is the same as a Picard character $\Lambda : G \rightarrow \text{Pic}^{\mathbb{Z}/2}(\mathbf{k})$, i.e., a family of super-lines $\Lambda_g, g \in G$, equipped with compatible multiplicativity and unit isomorphisms

$$\nu_{g,h} : \Lambda_g \otimes \Lambda_h \longrightarrow \Lambda_{gh}, \quad \nu_1 : \Lambda_1 \longrightarrow \mathbf{k},$$

see Definition 3.1.3. In particular, G acquires a $\mathbb{Z}/2$ -grading

$$G \longrightarrow \mathbb{Z}/2 \quad g \longmapsto \bar{g} := \deg(\Lambda_g).$$

Up to non-canonical equivalence of 2-representations, we may assume that $\Lambda_g = \Pi^{\bar{g}} \mathbf{k}$, that $\nu_1 = \text{Id}$ and that the $\nu_{g,h}$ are multiplication by the values $c(g,h) \in \mathbf{k}^*$ of a normalized 2-cocycle of G . We will refer to a 2-representation in this form as the *cocycle super 2-representation* ϱ_c .

Note the particular case $G = S_n$. In this case the sign homomorphism $\text{sgn} : S_n \rightarrow \{\pm 1\}$ can be seen as a $\mathbb{Z}/2$ -grading

$$S_n \longrightarrow \mathbb{Z}/2, \quad \sigma \longmapsto \bar{\sigma}, \quad (-1)^{\bar{\sigma}} = \text{sgn}(\sigma).$$

(3.4.3) Definition. The sign super 2-representation ϱ_{sgn} of S_n is the cocycle super 2-representation corresponding to the central extension \tilde{S}_n of S_n and to the $\mathbb{Z}/2$ -grading corresponding to the sign character, i.e.,

$$\Lambda_\sigma := \Pi^{\bar{\sigma}} L_\sigma,$$

and $\nu_{g,h}$ is defined by multiplication in \tilde{S}_n (see (3.2.5) ff.).

(3.4.4) Characters of cocycle super 2-representations. We now find the categorical character and the 2-character of a cocycle 2-representations ϱ_c as in Example (3.4.2). For a 2-representation ϱ of a group G on a linear category \mathcal{V} , its categorical character is the family of \mathbf{k} -vector spaces

$$X_\varrho(g) = \mathcal{N}\text{at}(\text{Id}_{\mathcal{V}}, \varrho(g)), \quad g \in G,$$

see [GK07]. These vector spaces form a conjugation invariant vector bundle on the discrete set G , i.e., we have isomorphisms

$$\psi_s : X_\varrho(g) \xrightarrow{\cong} X_\varrho(sgs^{-1}),$$

compatible with compositions of the h 's, see [GK07, Prop 4.10]. These constructions are extended to the superlinear case as follows:

(3.4.5) Definition. Let ϱ be a super-linear 2-representation. Then the categorical (super-) character of ϱ is the super-vector space

$$sX_\varrho^\bullet(g) = s\mathcal{N}\text{at}_\bullet(\text{Id}_{\mathcal{V}}, \varrho(g)).$$

To define the action of ψ_s on sX_ϱ^\bullet , we note that superlinearity of ρ makes Π an equivariant functor whose flip isomorphisms

$$\tau_s : \rho(s)\Pi \xrightarrow{\cong} \Pi\rho(s)$$

are given by the superlinearity data of the $\rho(g)$. Suppressing ϱ from the notation, ψ_s sends $\xi \in sX^\bullet(g)$ to the composite⁴

$$(\Pi\phi_{s,g,s^{-1}}) \circ (\tau_s g s^{-1}) \circ (s\xi s^{-1}) \circ \phi_{s,s^{-1}}^{-1} \circ \phi_1^{-1}.$$

Assuming all the $sX_\varrho^\bullet(g)$ are finite-dimensional, the 2-character of a super-linear 2-representation ϱ is defined as

$$\chi_\varrho(g, h) = \text{tr}\{\psi_g(h) : sX_\varrho^\bullet(g) \rightarrow sX_\varrho^\bullet(g)\}, \quad gh = hg.$$

This is a \mathbf{k}^2 -valued function, defined on pairs of commuting elements of G .

Let now c be a cocycle on a $\mathbb{Z}/2$ -graded group G , and let

$$1 \rightarrow \mathbf{k}^* \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

⁴This should be compared to the string diagram at the beginning of Section 5.3 in [?].

be the central extension classified by c . For a pair of commuting elements $g, h \in G$, we let

$$\epsilon(g, h) := \frac{c(g, h)}{c(h, g)}$$

and

$$e(g, h) := (-1)^{\deg(g)\deg(h)} \frac{c(g, h)}{c(h, g)}.$$

In terms of the group extension, $\epsilon(g, h)$ is the *symbol* of h and g in the sense of cf [Mil71, §8]. More precisely, for any lifts \tilde{g} of g and \tilde{h} of h , we have

$$\epsilon = \tilde{h}^{-1} \tilde{g}^{-1} \tilde{h} \tilde{g}.$$

We will refer to $e(g, h)$ as the *supersymbol* of g and h .

(3.4.6) Proposition. The 2-character of ϱ_c is given by

$$\chi_{\varrho_c}(g, h) = \begin{cases} (e(g, h) \mid 0) & \text{if } g \text{ is even and} \\ (0 \mid e(g, h)) & \text{if } g \text{ is odd.} \end{cases}$$

Proof. By definition of ϱ_c , its categorical character is given by

$$sX_{\varrho_c}^\bullet(g) = \Lambda_g = \Pi^{\bar{g}} L_g.$$

So,

$$\chi_{\varrho_c}(g, g) = \begin{cases} (\psi_h \mid 0) & \text{if } g \text{ is even and} \\ (0 \mid \psi_h) & \text{if } g \text{ is odd.} \end{cases}$$

For even g , the map ψ_h is the scalar by which the conjugation by h acts on the 1-dimensional vector space L_g . From the definition (3.2.5) of L_g we see that $\psi_g(h)$ describes the way by which the conjugation by (any lift of) h in \tilde{G} acts on the set $\varpi^{-1}(g)$. Hence $\psi_g(h) = \epsilon(g, h)$, as claimed. For odd g , the definition of ψ_h in Definition (3.4.5) includes the flip map τ_h . By the Koszul sign rule in (1.6.3), τ_h is multiplication with $(-1)^{\bar{h}}$. The rest is as in the even case (compare [Gan, Lem.6.1]). \square

(3.4.7) The character of the sign super 2-representation. In the case of \tilde{S}_n , the symbol $\epsilon(\sigma, \tau)$ was calculated by Dijkgraaf in [Di99]. Note that in Dijkgraaf's calculation, Formulas (3.20)ff. should include the sign

$$\epsilon(x_n, g') = (-1)^{n|g'|}.$$

With this correction, Dijkgraaf's result allows us to calculate the 2-character of ϱ_{sgn} .

(3.4.8) Definition. Let H be a finite group, and let $\varrho: H \rightarrow GL(V_\bullet)$ be a superrepresentation of H . Then we have the representation

$$\varrho \wr \text{sgn}: H \wr S_n \longrightarrow GL(V_\bullet^{\otimes n}),$$

where H^n acts by $\varrho^{\otimes n}$, and S_n permutes the tensor factors, adding a sign. Further, for even k , we let ϱ_{spin} be the $(0|1)$ -dimensional superrepresentation of $\mathbb{Z}/k\mathbb{Z}$ with character $(0|\chi_{\text{spin}})$, where

$$\chi_{\text{spin}}: \mathbb{Z}/k\mathbb{Z} \longrightarrow \{\pm 1\}$$

is the unique non-trivial map.

(3.4.9) Theorem. Let $\sigma \in S_n$ be classified by the partition $n = \sum_{k \geq 1} kN_k$. Then the supersymbol $e(\sigma, \tau)$, viewed as (one-dimensional) superrepresentation of the centralizer

$$C_\sigma \cong \prod_{k \geq 1} \mathbb{Z}/k\mathbb{Z} \wr S_{N_k},$$

equals

$$e(\sigma, \tau) = \bigotimes_{k \text{ even}} \chi_{\text{spin}} \wr \text{sgn}_{S_{N_k}}.$$

Together with Proposition (3.4.6) this determines the 2-character χ_{sgn} of ϱ_{sgn} .

(3.5) Equivariant objects. Throughout this section, we assume the characteristic of \mathbf{k} to be zero. Let ϱ be a super-2-representation of G on \mathcal{V} . Consider the category \mathcal{V}^G of equivariant objects in \mathcal{V} . If Π is G -equivariant, then the category \mathcal{V}^G is itself a superlinear category, with

$$\Pi(V, \{\epsilon_g\}) := (\Pi(V), ((\tau_g)_V) \circ (\Pi\epsilon_g)).$$

Here $\tau_g: \Pi g \Rightarrow g\Pi$ is the natural isomorphism making $\rho(g)$ a superlinear functor. The graded extension $(\mathcal{V}^G)_\bullet$ of \mathcal{V}^G is canonically identified with the full subcategory of $(\mathcal{V}_\bullet)^G$ whose objects have only even structure maps.

(3.5.1) Example. Let $\mathcal{V} = \text{SVect}^f$, and let G be a $\mathbb{Z}/2$ -graded group acting on \mathcal{V} via ϱ_c as in Example (2.5.2). Corresponding to the cocycle c , we have the twisted group superalgebra $\mathbf{k}[G^{\text{op}}]^c$. This is the associative \mathbf{k} -algebra generated by $\{e_g \mid g \in G\}$ with multiplication

$$(3.5.2) \quad e_g \cdot e_h = c(g, h)e_{hg}.$$

It is made into a superalgebra, i.e., equipped with the $\mathbb{Z}/2$ -grading, by putting $\deg(e_g) = \bar{g}$. The category of equivariant objects, \mathcal{V}^G , is equivalent, as a superlinear category, to the category $\mathbf{k}[G^{\text{op}}]^c\text{-Smod}^f$ of finite-dimensional left $\mathbf{k}[G^{\text{op}}]^c$ -supermodules and even morphisms between them. Note that the action of an odd element g on ΠV acquires the sign $\tau_g = -1$, as it should for left supermodules.

Applying⁵ [Gan, Thm.5.13], we obtain isomorphisms

$$(3.5.3) \quad \text{sCenter}(\mathcal{V}^G) \cong \bigoplus_{g \in G} sX_{\varrho}(g)^{C_g}.$$

and

$$(3.5.4) \quad s\text{Tr}(\Pi, \text{Id}) \cong \bigoplus_{g \in G} s\text{Tr}((\Pi, \text{Id})\varrho(g))^{C_g}.$$

For the remainder of this Section, we specialize to the situation of Example (2.5.2). So, we have a $\mathbb{Z}/2$ -graded group G , acting on SVect^f via a cocycle c .

Then the category $(\text{SVect}^f)^G$ is semisimple with finitely many isomorphism classes of simple objects.

(3.5.5) Definition. A conjugacy class $[g] \subseteq G$ is called *c-regular* if

$$(\forall h \in C_g) \quad (c(g, h) = c(h, g)).$$

⁵The theorem is applied inside the 2-category of super-linear categories, super-linear functors and super-natural transformations.

The following Corollary of (3.5.3) and (3.5.4) is a generalization of [BK, Lem 2.11].

(3.5.6) Corollary. The number of isomorphism classes of irreducible objects in $\text{Smod-}\mathbf{k}[G]^c$, up to shift of grading, equals the number of even c -regular conjugacy classes in G . Among these isomorphism classes, the number of self-associate ones equals the number of odd c -regular conjugacy classes in G .

PROOF : By (1.6.1)(c), the dimension of (3.5.3) counts the number of isomorphism classes of irreducible objects in \mathcal{V}^G , up to shift of grading, while the dimension of (3.5.4) counts the number of self-associate irreducible objects. For even g , the centralizer C_g acts by $\epsilon(g, -)$ on the one dimensional vectorspace $sX_{\theta_c}(g)$, and the g th summand of (2.6.4) is $s\text{Tr}(\Pi, \text{Id}) = 0$. Let g be odd. Then $sX_{\theta_c}(g) = 0$, while the g th summand of (3.5.4) equals $s\text{Tr}(\text{Id}, -\text{Id}) \cong \mathbf{k}$. By (1.6.1)(c), the action of $h \in C_g$ differs by the sign $(-1)^h$ from that in (3.5.6). In other words, h acts again by multiplication with $\epsilon(g, h)$. \square

We now specialize to the case of the symmetric group S_n .

(3.5.7) Theorem (Schur) [Sch11, IV]. An even conjugacy class $[\sigma]$ of S_n is c -regular if and only if all cycles of σ have odd length. An odd $[\sigma]$ is c -regular if and only if σ possesses no two cycles of equal length ≥ 1 .

(3.5.8) Corollary (compare [Kle05, Thm. 22.2.1]). The number of isomorphism classes of irreducible objects of $\text{Smod-}\mathbf{k}[S_n]^c$, up to shift of grading, equals the number of strict partitions of n . Among these, the number of absolutely irreducibles is equal to the number of even strict partitions, while the number of self-associates is equal to the number of odd strict partitions.

Proof. By Euler's theorem, the number of strict partitions of n equals the number of partitions of n into odd summands. The Corollary follows immediately. \square

(3.6) Exterior powers of categories. We now modify the naive definition of the exterior powers by incorporating the super point of view.

(3.6.1) Definition. Let \mathcal{V} be a superlinear category. The n th exterior power of \mathcal{V} is defined as

$$\Lambda^n(\mathcal{V}) = (\mathcal{V}^{\boxtimes_s n} \boxtimes_s \text{Sgn})^{S_n}.$$

If \mathcal{V} is an abelian superlinear category and $\mathcal{V}^{\hat{\boxtimes}_s n}$ exists, or if \mathcal{V} is a 2-periodic perfect dg-category, we define

$$\hat{\Lambda}^n(\mathcal{V}) = (\mathcal{V}^{\hat{\boxtimes}_s n} \boxtimes_s \text{Sgn})^{S_n}.$$

There is no need to complete the super tensorproduct with Sgn .

Explicitly, the exterior power $\Lambda^n(\mathcal{V})$ has, as objects, data consisting of:

- (a) An object $V \in \mathcal{V}^{\boxtimes_s n}$;
- (b) Isomorphisms

$$s_i : V \rightarrow \Pi\sigma_i^*(V), \quad i = 1, \dots, n-1,$$

satisfying the following conditions:

(c) For any $i = 1, \dots, n-1$, the composition

$$V \xrightarrow{s_i} \Pi\sigma_i^*(V) \xrightarrow{\Pi\sigma_i^*(s_i)} \Pi^2\sigma_i^*\sigma_i^*V \xrightarrow{\simeq} \Pi^2V = V$$

is equal to $(-\text{Id}_V)$.

(d) For any $i = 1, \dots, n-2$ the two compositions

$$V \xrightarrow{s_i} \Pi\sigma_i^*V \xrightarrow{\Pi\sigma_i^*(s_{i+1})} \Pi^2\sigma_i^*\sigma_{i+1}^*V \xrightarrow{\Pi^2\sigma_i^*\sigma_{i+1}^*(s_i)} \Pi^3\sigma_i^*\sigma_{i+1}^*\sigma_i^*V,$$

$$V \xrightarrow{s_{i+1}} \Pi\sigma_{i+1}^*V \xrightarrow{\Pi\sigma_{i+1}^*(s_i)} \Pi^2\sigma_{i+1}^*\sigma_i^*V \xrightarrow{\Pi^2\sigma_{i+1}^*\sigma_i^*(s_{i+1})} \Pi^3\sigma_{i+1}^*\sigma_i^*\sigma_{i+1}^*V$$

become equal after identifying the rightmost terms with

$$\Pi(\sigma_i\sigma_{i+1}\sigma_i)^*V = \Pi(\sigma_{i+1}\sigma_i\sigma_{i+1})^*V.$$

(e) For any i, j such that $|i-j| \geq 2$ the two compositions

$$V \xrightarrow{s_i} \Pi\sigma_i^*V \xrightarrow{\Pi\sigma_i^*(s_j)} \Pi^2\sigma_i^*\sigma_j^*V, \quad V \xrightarrow{s_j} \Pi\sigma_j^*V \xrightarrow{\Pi\sigma_j^*(s_i)} \Pi^2\sigma_j^*\sigma_i^*V$$

differ by sign, after identifying the rightmost terms with

$$(\sigma_i\sigma_j)^*V = (\sigma_j\sigma_i)^*V.$$

Morphisms are morphisms $V \rightarrow V'$ in $V^{\boxtimes n}$ commuting with the s_i .

Thus the difference with Λ_{naive}^n is that the s_i are assumed to be odd morphisms.

(3.6.2) Examples. (a) We have

$$\Lambda^n(\text{SVect}^f) = \text{Smod}^f\text{-}\mathbf{k}[S_n]^c.$$

In particular,

$$\text{rk } K(\Lambda^n(\text{SVect}^f)) = s(n).$$

(b) Let A be a finite-dimensional \mathbf{k} -superalgebra. We have then the crossed product superalgebra $A^{\otimes n} \otimes \mathbf{k}[S_n]^c$ generated by $A^{\otimes n}$ and odd generators s_i , $i = 1, \dots, n-1$ which satisfy the relations (2.4.1) together with the commutation relations

$$s_i \cdot a = a \cdot \sigma_i(a), \quad a \in A^{\otimes n}.$$

We have then

$$\widehat{\Lambda}^n(\text{Smod}^f\text{-}A) \simeq \text{Smod}^f\text{-}(A^{\otimes n} \otimes \mathbf{k}[S_n]^c).$$

4. THE CATEGORICAL KOSZUL COMPLEXES

(4.1) Statement of the result. Let \mathcal{V} be a superlinear category. In this section we assume one of the following:

- (a) \mathcal{V} is abelian, and $\mathcal{V}^{\widehat{\boxtimes}_s n}$ exists for every n .
- (b) \mathcal{V} is a 2-periodic perfect dg-category.

Under these assumptions, we have the completed symmetric and exterior powers $\widehat{\text{Sym}}^n \mathcal{V}$, $\widehat{\Lambda}^n \mathcal{V}$, as well as their products

$$(4.1.1) \quad \widehat{\Lambda}^p \mathcal{V} \widehat{\boxtimes}_s \widehat{\text{Sym}}^q \mathcal{V}, \quad p, q \geq 0.$$

The products in (4.1.1) are defined directly as consisting of objects of $V^{\widehat{\boxtimes}_s(p+q)}$ with two types of equivariance data commuting with each other: straight equivariance (2.2.1) with respect to $S_q \subset S_{p+q}$ permuting the last q factors, and “twisted” equivariance (3.6.1) with respect to $S_p \subset S_{p+q}$ permuting the first p factors. These products are again abelian superlinear in the case (a) and 2-periodic perfect in the case (b), so their Grothendieck groups are defined.

(4.1.2) Theorem. (a) There exist sequences of functors (categorical Koszul complexes)

$$0 \longrightarrow \widehat{\Lambda}^n \mathcal{V} \xrightarrow{D} (\widehat{\Lambda}^{n-1} \mathcal{V}) \widehat{\boxtimes}_s \mathcal{V} \xrightarrow{D} (\widehat{\Lambda}^{n-2} \mathcal{V}) \widehat{\boxtimes}_s (\widehat{\text{Sym}}^2 \mathcal{V}) \xrightarrow{D} \dots \xrightarrow{D} \widehat{\text{Sym}}^n \mathcal{V} \longrightarrow 0$$

$$0 \longrightarrow \widehat{\text{Sym}}^n \mathcal{V} \xrightarrow{\Delta} (\widehat{\text{Sym}}^{n-1} \mathcal{V}) \widehat{\boxtimes}_s \mathcal{V} \xrightarrow{\Delta} (\widehat{\text{Sym}}^{n-2} \mathcal{V}) \widehat{\boxtimes}_s (\widehat{\Lambda}^2 \mathcal{V}) \xrightarrow{\Delta} \dots \xrightarrow{\Delta} \widehat{\Lambda}^n \mathcal{V} \longrightarrow 0$$

(b) These functors are exact, if \mathcal{V} is abelian, and pre-exact, if \mathcal{V} is perfect. On the level of Grothendieck groups, these functors give rise to chain complexes:

$$D_*^2 = 0, \quad \Delta_*^2 = 0.$$

(c) The induced complexes of complexified Grothendieck groups are acyclic.

If the product maps

$$\otimes: K_{\mathbb{C}}^{\bullet}(\widehat{\Lambda}^p \mathcal{V}) \otimes K_{\mathbb{C}}^{\bullet}(\widehat{\text{Sym}}^q \mathcal{V}) \longrightarrow K_{\mathbb{C}}^{\bullet}(\widehat{\Lambda}^p \mathcal{V} \widehat{\boxtimes}_s \widehat{\text{Sym}}^q \mathcal{V})$$

of (1.7.3) are isomorphisms, then the complexes of the Grothendieck groups in (4.1.2) look even more like Koszul complexes.

(4.1.3) Example. Let A be a finite-dimensional \mathbf{k} -superalgebra, and \mathcal{V} be the abelian category $\text{Smod}^f\text{-}A$. Then

$$\widehat{\text{Sym}}^n(\mathcal{V}) = \text{Smod}^f\text{-}(A^{\otimes n}[S_n]), \quad \widehat{\Lambda}^n(\mathcal{V}) = \text{Smod}^f\text{-}(A^{\otimes n} \otimes \mathbf{k}[S_n]^c),$$

see Examples (1.6.7), (2.2.8)(b) and (3.6.2)(b). In this case the homomorphisms (1.7.3) are isomorphisms (see Example (1.7.4)(c)), and we get the equality

$$\left(\sum_{n \geq 0} \dim K_{\mathbb{C}}^{\bullet}(\text{Smod}^f\text{-}A^{\otimes n}[S_n]) q^n \right) \cdot \left(\sum_{n \geq 0} \dim K_{\mathbb{C}}^{\bullet}(\text{Smod}^f\text{-}(A^{\otimes n} \otimes \mathbf{k}[S_n]^c)) (-q)^n \right) = 1.$$

(4.2) Partial symmetrization. Let G be a finite group acting on a linear category \mathcal{W} , so we have functors $\varrho(g) : \mathcal{W} \rightarrow \mathcal{W}$ as in (2.1). Let H be a subgroup of G . Then the forgetful functor from the category of G -equivariant to that of H -equivariant objects has a left adjoint,

$$\begin{aligned} I_H^G : \mathcal{W}^H &\longrightarrow \mathcal{W}^G \\ W &\longmapsto \bigoplus_{[g] \in G/H} \varrho(g)(W), \end{aligned}$$

called ‘*partial symmetrization*’. Here g runs over representative classes of right cosets of G by H , and the direct sum carries the obvious G -equivariant structure.

(4.2.1) Examples. (a) Suppose that the G -action on \mathcal{W} is trivial. Then W is a representation of H in \mathcal{W} , and $I_H^G(W) = \text{Ind}_H^G(W)$ is the induced representation.

(b) Let $G = S_{q+1}$ be the group of permutations of $\{0, 1, \dots, q\}$, and $H = S_q$ be the group of permutations of $\{1, \dots, q\}$. Then a set of representatives for S_{q+1}/S_q is provided by the transpositions

$$\text{Id}, (0, 1), (0, 2), \dots, (0, q),$$

so for an S_q -equivariant object W its partial symmetrization is given by

$$I_{S_q}^{S_{q+1}}(W) = W \oplus \bigoplus_{i=1}^q \varrho((0, i))(W).$$

(c) Let $H = 1$ is the trivial group. The functor I_1^G plays a crucial role in the proof of the main theorem in [Gan], where I_1^G is denoted A' .

The following is obvious from the definition of $I_H^G(W)$ as a direct sum.

(4.2.2) Proposition. If \mathcal{W} is abelian, then I_H^G is exact. If \mathcal{W} is a perfect dg-category, then I_H^G is pre-exact.

(4.3) The functors $D_{p,q}$ and $\Delta_{p,q}$. Denote by

$$J_p : \widehat{\Lambda}^p \mathcal{V} \longrightarrow (\widehat{\Lambda}^{p-1} \mathcal{V}) \widehat{\boxtimes}_s \mathcal{V}$$

the functor of forgetting the part of the twisted equivariance data not pertaining to S_{p-1} . In other words, an object of $\widehat{\Lambda}^p(\mathcal{V})$ is an object $V \in \mathcal{V}^{\widehat{\boxtimes}_{s^p}}$ with isomorphisms $s_i : V \rightarrow \Pi \sigma_i^*(V)$, $i = 1, \dots, p-1$, satisfying the conditions in (3.6.1). By considering only s_1, \dots, s_{p-2} , we get an object of $\widehat{\Lambda}^{p-1}(\mathcal{V}) \widehat{\boxtimes}_s \mathcal{V}$, which we denote $J_p(V)$. We now define the functor $D_{p,q}$ to be the composition

$$\widehat{\Lambda}^p \mathcal{V} \widehat{\boxtimes}_s \widehat{\text{Sym}}^q \mathcal{V} \xrightarrow{J_p \widehat{\boxtimes}_s \text{Id}} (\widehat{\Lambda}^{p-1} \mathcal{V}) \widehat{\boxtimes}_s \mathcal{V} \widehat{\boxtimes}_s (\widehat{\text{Sym}}^q \mathcal{V}) \xrightarrow{\text{Id} \widehat{\boxtimes}_s I_{S_q}^{S_{q+1}}} (\widehat{\Lambda}^{p-1} \mathcal{V}) \widehat{\boxtimes}_s (\widehat{\text{Sym}}^{q+1} \mathcal{V}).$$

Similarly, we have the functor

$$\widetilde{J}_q : \widehat{\text{Sym}}^q \mathcal{V} \longrightarrow \mathcal{V} \widehat{\boxtimes}_s \widehat{\text{Sym}}^{q-1} \mathcal{V}$$

forgetting the part of equivariance structure not pertaining to S_{q-1} . Again, we have the partial symmetrization functor

$$\tilde{I}_{S_{p-1}}^{S_p} : \widehat{\Lambda}^p(\mathcal{V}) \widehat{\boxtimes}_s \mathcal{V} = (\mathcal{V}^{\widehat{\boxtimes}_s(p+1)} \boxtimes_s \text{Sgn})^{S_p} \rightarrow (\mathcal{V}^{\widehat{\boxtimes}_s(p+1)} \boxtimes_s \text{Sgn})^{S_{p+1}} = \widehat{\Lambda}^{p+1}(\mathcal{V}).$$

Explicitly, for an object W of $\widehat{\Lambda}^p(\mathcal{V}) \widehat{\boxtimes}_s \mathcal{V}$ we have

$$\tilde{I}_{S_p}^{S_{p+1}}(W) = W \oplus \bigoplus_{i=1}^p \Pi \varrho((i, p+1))(W),$$

where ρ is the permutation 2-representation. We define the functor $\Delta_{p,q}$ to be the composition

$$\widehat{\Lambda}^p \mathcal{V} \widehat{\boxtimes}_s \widehat{\text{Sym}}^q \mathcal{V} \xrightarrow{\tilde{J}_q \widehat{\boxtimes}_s \text{Id}} (\widehat{\Lambda}^p \mathcal{V}) \widehat{\boxtimes}_s \mathcal{V} \widehat{\boxtimes}_s (\widehat{\text{Sym}}^{q-1} \mathcal{V}) \xrightarrow{\text{Id} \widehat{\boxtimes}_s \tilde{I}_{S_{p-1}}^{S_p}} (\widehat{\Lambda}^{p+1} \mathcal{V}) \widehat{\boxtimes}_s (\widehat{\text{Sym}}^{q-1} \mathcal{V}).$$

It is clear that the $D_{p,q}$ and $\Delta_{p,q}$ are exact if \mathcal{V} is abelian and pre-exact if \mathcal{V} is perfect. So they induce morphisms $(D_{p,q})_*$ and $(\Delta_{p,q})_*$ of the Grothendieck groups.

(4.3.1) Proposition. We have the identities:

$$\begin{aligned} (D_{p-1,q+1})_*(D_{p,q})_* &= 0, \\ (\Delta_{p-1,q+1})_*(\Delta_{p,q})_* &= 0, \quad \text{and} \\ (\Delta_{p-1,q+1})_*(D_{p,q})_* + (D_{p+1,q-1})_*(\Delta_{p,q})_* &= (p+q) \cdot \text{Id}. \end{aligned}$$

Thus the first two series of equalities mean that we have chain complexes, while the third series implies that the complexes become acyclic after tensoring with \mathbb{C} . Indeed, they mean that Δ_* is a contracting homotopy for D_* and vice versa: the isomorphism of the n th complex given by multiplication by $n = p + q$ is homotopic to zero. Thus Proposition 4.3.1 implies Theorem 4.1.2.

Proof of (4.3.1): We start with the first identity. It is enough to consider $p = 2$, the first $p - 2$ arguments being immaterial. The composition

$$(\widehat{\Lambda}^2 \mathcal{V}) \widehat{\boxtimes}_s (\widehat{\text{Sym}}^q \mathcal{V}) \xrightarrow{D_{2,q}} \mathcal{V} \widehat{\boxtimes}_s (\widehat{\text{Sym}}^{q+1} \mathcal{V}) \xrightarrow{D_{1,q+1}} \widehat{\text{Sym}}^{q+2} \mathcal{V}$$

is just the partial symmetrization $I_{S_q}^{S_{q+2}}$. Here S_q acts on $\{1, \dots, q+2\}$ by fixing the first two elements. Hence the elements of a coset σS_q are exactly those elements $\tau \in S_{q+2}$ with $\tau(1) = \sigma(1)$ and $\tau(2) = \sigma(2)$. So, any system of coset-representatives $\{\sigma_{i,j}\}$ may be labeled by ordered pairs i, j in $\{1, \dots, q+2\}$, with $\sigma_{i,j}(1) = i$ and $\sigma_{i,j}(2) = j$. Note that we can choose these representatives in such a way that we have

$$\sigma_{i,j} = \sigma_{j,i} \circ (1, 2).$$

Let V be an object of $(\widehat{\Lambda}^2 \mathcal{V}) \widehat{\boxtimes}_s (\widehat{\text{Sym}}^2 \mathcal{V})$. Then V is twisted equivariant in first two factors, and we have

$$\varrho((1, 2))(V) \simeq \Pi V,$$

and more generally,

$$\varrho(\sigma_{i,j})(V) \cong \Pi \varrho(\sigma_{j,i}(V)).$$

Hence their classes in the K -group cancel, and

$$(D_{1,q+1})_*(D_{2,q})_*\langle V \rangle = 0.$$

The second identity, $(\Delta_{1,q+1})_*(\Delta_{2,q})_* = 0$, is proved in a similar way.

We now prove the third identity. Let V be an object of $\widehat{\Lambda}^p(\mathcal{V}) \widehat{\boxtimes}_s \widehat{\text{Sym}}^q(\mathcal{V})$. As in (4.1), we view V as an object of $\mathcal{V}^{\widehat{\boxtimes}_s(p+q)}$ with two types of equivariance data. Then

$$\Delta_{p-1,q+1}(D_{p,q}(V)) \cong \left(V \oplus \bigoplus_{i=1}^{p-1} \Pi(\varrho(i,p)V) \right) \oplus \bigoplus_{j=1}^q \left(\varrho(p,p+j)V \oplus \bigoplus_{i=1}^{p-1} \Pi(\varrho(i,p,p+j)V) \right).$$

By virtue of the two kinds of equivariance data of V , the first p summands each are isomorphic to V , each of the next q summands is isomorphic to one of the form $\varrho(p,p+j,p+1)$, and the remaining $q(p-1)$ summands are isomorphic to ones of the form $\varrho(i,p+j,p+1)$. Here the last step used that

$$(i,p,p+j) = (i,p+j,p+1) \circ (p+1,p+j)(i,p).$$

Likewise, $D_{p+1,q-1}(\Delta_{p,q}(V))$ is a direct sum of $(p+1)q$ terms, q of which are isomorphic to V . The remaining pq terms are $\Pi(\varrho(i,p+j,p+1))$ with $1 \leq i \leq p$ and $1 \leq j \leq q$, pairing up with the corresponding terms in $\Delta_{p-1,q+1}D_{p,q}(V)$ to cancel in the Grothendieck group. \square

5. EXAMPLES FROM REPRESENTATION THEORY

Let \mathcal{V} be a superlinear category of one of the types considered in (4.1). The spaces

$$(5.1) \quad \bigoplus_{n \geq 0} K_{\mathbb{C}}^{\bullet}(\widehat{\text{Sym}}^n \mathcal{V}) \quad \text{and} \quad \bigoplus_{n \geq 0} K_{\mathbb{C}}^{\bullet}(\widehat{\Lambda}^n \mathcal{V})$$

can be thought as analogues of the bosonic and fermionic Fock spaces. Several recent papers realize these spaces as basic representations of some infinite-dimensional Lie algebras. In each of these situation the first space corresponds to the “untwisted” case, while the second one corresponds to an appropriately “twisted” case. In this section we recall some examples and give their interpretation from our point of view.

(5.2) Wedge and spin representations. (a) Take $\mathcal{V} = \text{SVect}^f$ to be the category of finite-dimensional super \mathbf{k} -vector spaces. Then $\text{Sym}^n \mathcal{V} = \text{Rep}_{\mathbf{k}}(S_n)$ is the category of finite-dimensional $\mathbb{Z}/2$ -graded representations of S_n over \mathbf{k} . Here it is not necessary to complete symmetric and exterior powers. There are no self-associate irreducible objects in $\text{Rep}_{\mathbf{k}}(S_n)$, and

$$K_{\mathbb{C}}^{\bullet}(\text{Sym}^n \mathcal{V}) = K_{\mathbb{C}}^0(\text{Sym}^n \mathcal{V})$$

is the complexified representation ring of S_n . Denote this ring $\mathcal{R}_{\mathbf{k}}(S_n)$.

Assume $\text{char}(\mathbf{k}) = 0$. Then a basis of $\mathcal{R}_{\mathbf{k}}(S_n)$ is labelled by partitions $\alpha = (\alpha_1 \geq \dots \geq \alpha_l)$ of n which label irreducible representations. In fact, we have an identification

$$\bigoplus_{n \geq 0} \mathcal{R}_{\mathbf{k}}(S_n) \simeq \mathfrak{F}$$

where the right-hand side is the basic projective representation of the Lie algebra

$$\mathfrak{gl}(\infty) := \{(a_{i,j})_{i,j \in \mathbb{Z}} \mid \text{almost all } a_{i,j} \text{ equal } 0\},$$

see [KR87, Lecture 4] or [Kle05, Rem. 2.3.13]. Here \mathfrak{F} stands for “Fock”. To describe \mathfrak{F} , let $V = \mathbb{C}[t, t^{-1}]$ be the vector space of all finite formal Laurent polynomials over \mathbb{C} . Then \mathfrak{gl}_{∞} acts on V in the standard way: let $E_{i,j}$ be the matrix with only one entry, 1 in the spot (i, j) . Then $E_{i,j} t^k = \delta_{j,k} t^i$. The semi-infinite wedge $\Lambda^{\infty/2} V$ inherits a \mathfrak{gl}_{∞} -action from V , and \mathfrak{F} is the smallest subrepresentation containing the ‘*vacuum vector*’

$$1 \wedge t^{-1} \wedge t^{-2} \wedge t^{-3} \wedge \dots$$

A basis of \mathfrak{F} is given by semiinfinite wedge products of monomials in t :

$$t^{i_0} \wedge t^{i_1} \wedge t^{i_2} \wedge \dots, \quad i_0 > i_1 > \dots, \quad i_k = -k, \quad k \gg 0.$$

Now, given such a wedge, we form a partition

$$\alpha = (\alpha_0 \geq \alpha_1 \geq \dots), \quad \alpha_k = k + i_k,$$

and this gives a bijection between the bases of the two spaces. In fact, the Young graph of partitions and their inclusions is interpreted as the “crystal graph” of the wedge representation. It means that the edges of the Young graph (which correspond to adding/removing a node from a Young diagram) exactly describe the action of the E_{ij} on the basis wedge products).

(b) Keeping the assumption $\text{char}(\mathbf{k}) = 0$, we have the following companion description for the exterior powers:

$$\bigoplus_n K_{\mathbb{C}}^{\bullet}(\Lambda^n(\mathcal{V})) \simeq \Sigma,$$

where Σ is the basic “spin” projective representation of the Lie algebra $\mathfrak{o}(\infty)$. The direct sum on the left is the direct sum of the Grothendieck groups of the categories $\text{Smod}^f - \mathbf{k}[S_n]^c$ of projective super-representations of S_n . The representation Σ was defined in [KP81], while its interpretation in terms of projective representations of symmetric groups was studied in [Jin91], using the techniques of vertex operators.

(5.3) The Kac-Moody algebras $A_{p-1}^{(1)}$ and $A_{p-1}^{(2)}$. Consider now the same situation as in (5.2) except assume that $\text{char}(\mathbf{k}) = p > 0$.

Irreducible representations of S_n over \mathbf{k} are now labelled by p -regular partitions, i.e., partitions where no part is repeated p or more times, see [Kle05] for details. For example, 2-regular means strict. The spaces (5.1) in this case are identified with the fundamental (projective) representations of the Kac-Moody Lie algebras

$$(5.3.1) \quad A_{p-1}^{(1)} = \mathfrak{sl}_p(\mathbb{C}[t, t^{-1}]), \quad A_{p-1}^{(2)} = \{g(t) \in \mathfrak{sl}_p \mathbb{C}[t, t^{-1}] : g(t^{-1}) = g(t)^*\}.$$

Here $*$ means transposition. The symmetric powers in (5.1) make sense for all p , while the exterior powers are defined only for $p > 2$, in the same way as the Lie algebras in (5.3.1). See [Kle05] for more details and references.

(5.4) Twisted and untwisted Kac-Moody algebras of type ADE. Let $\text{char}(\mathbf{k}) = 0$ and let Γ be a finite group isomorphic to a subgroup of $SL_2(\mathbb{C})$. By the McKay correspondence, Γ corresponds to a root system of type ADE. Denote by \mathfrak{g}_{Γ} the corresponding semisimple split Lie algebra over \mathbb{C} . We have then the Kac-Moody algebras $\mathfrak{g}_{\Gamma}^{(1)}, \mathfrak{g}_{\Gamma}^{(2)}$ defined as in (5.3.1) but with the transposition replaced by the Cartan involution.

On the other hand, take $\mathcal{V} = \text{Rep}(\Gamma)$ to be the category of finite-dimensional representations of Γ . Then

$$\widehat{\text{Sym}}^n(\mathcal{V}) = \text{Rep}(S_n \wr \Gamma^n), \quad \widehat{\Lambda}^n(\mathcal{V}) = \text{Rep}_c(S_n \wr \Gamma^n)$$

are the categories of the all representations (resp. projective representations of a certain kind) of the wreath product $S_n \wr \Gamma^n$. In fact, since we work in characteristic zero, all the categories involved are semisimple and we can drop the hats over symmetric and exterior powers.

The Grothendieck groups of such representations were studied in [FJW00], [FJW03], whose results mean that

$$\bigoplus_{n \geq 0} K_{\mathbb{C}}^{\bullet}(\text{Sym}^n(\mathcal{V})) = \mathfrak{F}_{\Gamma}^{(1)}, \quad \bigoplus_{n \geq 0} K_{\mathbb{C}}^{\bullet}(\Lambda^n(\mathcal{V})) = \mathfrak{F}_{\Gamma}^{(2)}$$

are the fundamental representations of $\mathfrak{g}_{\Gamma}^{(1)}$ and $\mathfrak{g}_{\Gamma}^{(2)}$ respectively.

6. EFFECT ON CATEGORICAL CHARACTERS

Let ϱ be a linear 2-representation of G on $\mathcal{V} = \text{Vect}^m$. By [KV94], we may view ϱ as a matrix-representation: $\varrho(g)$ is described by an $m \times m$ -matrix $(V_{i,j})$ whose entries are vectorspaces, and the $\phi_{g,h}$ and ϕ_1 amount to matrices of isomorphisms between the relevant entries.

(6.1) Symmetric powers. The symmetric powers $\text{Sym}^n(\varrho)$ are again 2-representations of G , and their categorical characters are described as follows: for $g \in G$, we have an isomorphism of \mathbb{N} -graded C_g -representations

$$(6.1.1) \quad \bigoplus_{n \geq 0} X_{\text{Sym}^n(\varrho)}(g) q^n \cong \bigotimes_{k \geq 1} S_{q^k} (X_{\varrho}(g^k)^{\mathbb{Z}/k\mathbb{Z}}).$$

Here q is a dummy variable, and

$$S_q(V) = \bigoplus \text{sym}^m(V) q^m$$

is the total symmetric power of the vector space V .

Proof of (6.1.1). The n th tensor power $\mathcal{V}^{\boxtimes n}$ carries an action of $S_n \wr G$. The group G still acts diagonally on the category $\text{Sym}^n(\mathcal{V})$ of S_n -equivariant objects. By [Gan, Theorem 5.13], we have an isomorphism of vector spaces

$$X_{\text{Sym}^n(\varrho)}(g) \cong \left(\bigoplus_{\sigma \in S_n} X_{\varrho^{\boxtimes n}}(\sigma; g, \dots, g) \right)^{S_n}.$$

Fix $\sigma \in S_n$ and let n_k be the number of k -cycles in σ . Then the action of $(\sigma; g, \dots, g)$ on $\mathcal{V}^{\boxtimes n}$ is isomorphic to the (external) categorical tensor product of auto-equivalences $(c_k; g, \dots, g)$ of \mathcal{V}^k , where c_k acts as a cyclic permutation of the tensor factors. Since the categorical trace is multiplicative (see [Gan, Thm. 2.5]), we are reduced to calculating the trace of $(c_k; g, \dots, g)$. An examination of the diagonal entries of the relevant matrix yields an isomorphism

$$X_{\varrho^{\boxtimes k}}(c_k; g, \dots, g) \cong X_{\varrho}(g^k).$$

More precisely, both sides are isomorphic to

$$\bigoplus_{\underline{i}} V_{i_1, i_2} \otimes V_{i_2, i_3} \otimes \dots \otimes V_{i_k, i_1},$$

and $\langle c_k \rangle$ acts by cyclic permutation of the tensor factors. Taking fixed points under the action of the centralizer

$$C_{\sigma} \cong \prod_k S_{n_k} \wr (\mathbb{Z}/k\mathbb{Z}),$$

we obtain

$$X_{\varrho^{\boxtimes n}}(\sigma; g, \dots, g)^{C_{\sigma}} \cong \bigotimes_k \text{sym}^{n_k} (X_{\varrho}(g^k)^{\mathbb{Z}/k\mathbb{Z}}),$$

and hence

$$X_{\text{Sym}^n(\varrho)}(g) \cong \bigoplus_{n = \sum k n_k} \bigotimes_{k \geq 1} \text{sym}^{n_k} (X_{\varrho}(g^k)^{\mathbb{Z}/k\mathbb{Z}}).$$

This is the coefficient of q^n on the right-hand side of (6.1.1). □

(6.2) Exterior powers. Let now ϱ be a superlinear 2-representation of G on $\mathcal{V} = (\text{SVect}^f)^m$. Just as in the linear case, we may view ϱ as a matrix-representation: $\varrho(g)$ is described by an $m \times m$ -matrix of supervectorspaces, and the $\phi_{g,h}$ and ϕ_1 amount to even maps between such matrices. The formalism of [KV94] goes through in the super setting. For instance, composition of superlinear functors corresponds to matrix multiplication, with $+$ and \cdot replaced by \oplus and \otimes of supervectorspaces.

To calculate the categorical supercharacter of the total exterior power of ϱ , we apply the same argument as in the proof of (6.1.1), but this time with a different S_n -action. This yields the formula

$$(6.2.1) \quad \bigoplus_{n \geq 0} sX_{\Lambda^n(\varrho)}^\bullet(g) q^n \cong \bigotimes_{k \geq 1} S_{q^{2k-1}}(sX_\varrho^\bullet(g^{2k-1})^+) \otimes \Lambda_{q^{2k}}(sX_\varrho^{\bullet+\bar{1}}(g^{2k})^-).$$

For odd k , the cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts on $sX^\bullet(g^k)$ as in the proof of (6.1.1). For even k , the action acquires the sign χ_{spin} (see Definition (3.4.8)). The superscripts $+$ and $-$ indicate the invariant parts under these respective actions, and Λ_q denotes the total exterior power of supervectorspaces.

7. OPEN QUESTIONS

(7.1) Symmetric product orbifolds with discrete torsion. With the corrected ϵ of Section (3.4.7), Dijkgraaf's Formula [Di99, (4.7)] for the Hilbert space of the symmetric product orbifold with discrete torsion becomes

$$(7.1.1) \quad \mathcal{H}^c(S^N X) = \bigoplus_{\substack{\text{even } \{N_n\} \\ N = \sum n N_n}} \bigotimes_{n>0} (S^{N_{2n-1}} \mathcal{H}_{2n-1}^+ \otimes \Lambda^{N_{2n}} \mathcal{H}_{2n}^-) \oplus \bigoplus_{\substack{\text{odd } \{N_n\} \\ N = \sum n N_n}} \bigotimes_{n>0} \Lambda^{N_n} \mathcal{H}_n^+.$$

In other words, if the overall parity of σ is odd then long strings of odd length exhibit fermionic, not bosonic behaviour. There is no obvious way to summarize (7.1.1) into a product formula for all N , as in [DMVV97]. One would expect this to change when the super point of view is taken into account: assume that Dijkgraaf's Hilbert space \mathcal{H} is $\mathbb{Z}/2$ -graded by the Fermion number.

(7.1.2) Question. Can one incorporate the $\mathbb{Z}/2$ -grading of S_n and the Koszul sign

$$(-1)^{\deg(\sigma) \deg(\tau)}$$

into the CFT-interpretation [Di99, 4.2] such that (6.1.1) becomes

$$(7.1.3) \quad \mathcal{H}_\bullet^c(S^N X) = \bigoplus_{N = \sum n N_n} \bigotimes_{n>0} S^{N_{2n-1}} (\mathcal{H}_{\bullet, 2n-1}^+) \otimes \Lambda^{N_{2n}} (\mathcal{H}_{\bullet+1, 2n}^-) \quad ?$$

Here $\mathcal{H}_\bullet^c(S^N X)$ is the super-Hilbertspace of the symmetric product orbifold with discrete torsion.

Formula (7.1.3) is obtained by replacing ϵ with e in Dijkgraaf's argument. Note that with this adjustment, n -cycles behave as “bosons” if n is odd and as “fermions” if n is even, as postulated in [Di99, p.7], but the distinction between even and odd partitions in [Di99, (4.7)] has vanished.

Further, one would expect a Dijkgraaf-Moore-Verlinde-Verlinde type product formula for the super-Hilbertspace of the total symmetric product orbifold with discrete torsion:

$$\sum_{n \geq 0} \mathcal{H}_\bullet^c(S^n) q^n = \bigotimes_{k \geq 1} S_{q^{2k-1}} (\mathcal{H}_{\bullet, 2k-1}^+) \otimes \Lambda_{q^{2k}} (\mathcal{H}_{\bullet+1, 2k}^-).$$

Here q is a formal variable and S_q and Λ_q denote the total symmetric and exterior power of super vectorspaces.

(7.2) Operations on the Bondal-Larsen-Lunts ring. Our results suggest that the Grothendieck ring \mathcal{PT} of pretriangulated categories defined in [BLL] has well-defined non-linear operations Sym^n . The relations in \mathcal{PT} come from semi-orthogonal decomposition, so a study of these operations would need to address the following question.

(7.2.1) Question. How do the Sym^n interact with semi-orthogonal decompositions?

(7.2.2) Question. Is there a way to incorporate the super point of view and exterior powers into this picture?

It is natural to expect that the formalism for these operations is similar to that of a 2-special lambda ring, see [GK] or [Rez]. Given a group G , one can consider pretriangulated categories with G -action, so we get a kind of 2-representation ring \mathcal{PT}_G , which should also

have this kind of structure. Let c_k be a cyclic permutation of length k , and note that the categorical trace of the action of $(c_k; g, \dots, g)$ on X^k involves the fixed point set $(X^k)^{(c_k; g, \dots, g)}$. This can be identified with the fixed points $X^{g^k} \subseteq X$, who are in turn involved in the categorical trace $\mathrm{Tr}^\bullet(g^k)$. This observation motivates the following question.

(7.2.3) Question. Is it possible to generalize the discussion of Section 6 to a wider class of 2-representations?

(7.3) Elliptic genera. The expression $\bigotimes_{k \geq 1} S_{q^k}(-)$ in Formula (6.1.1) is familiar from the definition of the Witten genus. Formula (6.2.1) reminds of the “expansion at the other cusp of the equivariant signature of the loop space” in [HB92, p.82].

(7.3.1) Question. Can one use our constructions to define the Witten genus of X in terms of $D^b(X)$, the bounded derived category of coherent sheaves on X (or its pre-triangulated enhancement)?

Note that if we use the tensor structure on the derived category, then an affirmative answer to this question follows indirectly from results of Balmer [Bal05], who is able to recover the whole scheme X from $D^b(X)$ as a tensor triangulated category. However, there are examples of pairs X, X' of algebraic varieties (related by a birational “flop”) for which we have an equivalence $D^b(X) \simeq D^b(X')$ (not preserving tensor structures), as well as an equality of the Witten genera. See [Bri02, Tot00]. This suggest that there may be a relation between the (pre-)triangulated category structure on $D^b(X)$ and the Witten genus.

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